CS 468 (Spring 2013) — Discrete Differential Geometry

Lectures 4 and 5: Surfaces

Reminder: the differential of a function.

- The tangent space of $\mathbb{R}^n$ at p, denoted $T_p\mathbb{R}^n$. Tangent vectors of curves.
- The differential of $f : \mathbb{R}^n \to \mathbb{R}^m$ at p is the matrix $Df_p \in \mathbb{R}^{m\times n}$ with components $\frac{\partial f^i}{\partial x^j}$.
- Interpretation as a linear mapping $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$. Image of curves and their tangent vectors. Let $c : I \to \mathbb{R}^n$ be a curve with $c(0) = p$ and $\dot{c}(0) = X_p$. Then

$$\frac{d}{dt} f(c(t))\Big|_{t=0} = \left(\ldots, \sum_i \frac{\partial f^j}{\partial x^i} \circ c(t) \frac{dc^i(t)}{dt}\Big|_{t=0}, \ldots \right) = Df_p \cdot X_p$$

- The rank of Df_p. Injectivity and surjectivity.
- Qualitative picture of a map of locally constant rank. Let $f : \mathbb{R}^n \to \mathbb{R}^m$.
  - If Df_p is injective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n \leq m$ and we can "modify" f as follows: there exist smooth bijections with smooth inverses (a.k.a. diffeomorphisms) $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^m \to \mathbb{R}^m$ (actually defined on suitable open sets of Ω and $f(\Omega)$) so that the map $\tilde{f} := \psi \circ f \circ \phi^{-1}$ has the form
  $$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$
  for all $x := (x^1, \ldots, x^n)$ in the domain of ϕ.
  - If Df_p is surjective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n \geq m$ and a similar modification of f has the form
  $$\tilde{f}(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^1, \ldots, x^m)$$
  for all $x := (x^1, \ldots, x^n)$ in the domain of ϕ. Note that $\tilde{f}$ can be many-to-one since, for instance, we have $\tilde{f}^{-1}(0) = \{(0, \ldots, 0, x^{m+1}, \ldots, x^n) : x^i \in \mathbb{R} \text{ for each } i\}$.
  - If Df_p is bijective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n = m$ and a similar modification of f has the form
  $$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n)$$
  for all $x := (x^1, \ldots, x^n)$ in the domain of ϕ. Note that $\tilde{f}$ and thus f are locally bijective.
  - If Df_p has rank k for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $k \leq \min(n, m)$ and a similar modification of f has the form
  $$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0)$$
  for all $x := (x^1, \ldots, x^n)$ in the domain of ϕ.
- Proofs are based on the *inverse* and *implicit function theorems.*

  InvFT. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with Df_p bijective, then f is invertible on a neighbourhood of p. Note that Df_p is bijective at p if and only if $\det(Df_p) \neq 0$. This is an *open condition* so we actually obtain a stronger result than above.

  ImpFT. If $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $D_1F_{(p,q)}$ bijective and $F(p, q) = 0$, then the equation $F(x, y) = 0$ can be solved for points (x, y) near (p, q) in the following sense. There exists a function $g : \mathbb{R}^k \to \mathbb{R}^n$ defined in a neighbourhood of q giving us $y = g(x)$ for which $q = g(p)$ and also $F(x, g(x)) = 0$. Note that we can compute Dg_x in terms of $D_1F_{(x,g(x))}$ and $D_2F_{(x,g(x))}$. Example: $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Three kinds of surfaces.

- Common representations of surfaces in $\mathbb{R}^3$.
- Graphs of functions $f : \mathbb{R}^2 \to \mathbb{R}$. Examples: planes, upper hemisphere.
- Level sets of functions $F : \mathbb{R}^3 \to \mathbb{R}$. Examples: the whole sphere. Conic sections. Graphs as the zero level set of $F(x, y, z) := z - f(x, y)$. Writing a level set as a graph — when this is possible, and the relation to ImpFT.
- Parametric surfaces $\sigma : U \to \mathbb{R}^3$ where $U \subseteq \mathbb{R}^2$ is an open domain in the plane and $\sigma(u^1, u^2) := (\sigma^1(u^1, u^2), \sigma^2(u^1, u^2), \sigma^3(u^1, u^2))$. Examples: sphere, torus. Graphs as parametrized surfaces $(x, y) \mapsto (x, y, f(x, y))$. Relation with level sets: $F(\sigma(u)) = \mathit{const}$ for all $u \in U$.
- Suppose you come across a surface in $\mathbb{R}^3$, what representation do you choose to describe it mathematically? Each representation has its limitations.
  - Not every surface is a graph.
  - How do you find a level set function? Or if you know the level set function, how do you solve it? You have to solve equations! E.g. if $F(x, y, z) = 0$ you need to extract $z = g(x, y)$ with the property that $F(x, y, g(x, y)) = 0$.
  - In general only part of a surface can be nicely parametrized. Non-uniqueness.

The definition of a surface.

- We would like a definition of a surface that as independent of representation as possible. The method of choice is: *local parametrizations*.
- A a set of points $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$ there exists an open neighbourhood $V \subseteq \mathbb{R}^3$ containing p, an open neighbourhood $U \subseteq \mathbb{R}^2$ and a parametrization $\sigma : U \to V \cap S$ such that:
  1. $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ is differentiable (i.e. each $\sigma^i : U \to \mathbb{R}$ is a smooth function).
  2. σ is invertible (as a map from the parameter domain onto its image) with continuous inverse. I.e. there is a function $\sigma^{-1} : V \cap S \to U$ such that $\sigma \circ \sigma^{-1} = id_{V \cap S}$ and $\sigma^{-1} \cap \sigma = id_U$; and also σ^{-1} is the restriction to $V \cap S$ of a continuous function on an open neighbourhood $W \subseteq \mathbb{R}^3$ containing $V \cap S$ onto U.
  3. For every $q \in U$, the differential $D\sigma_q$ is injective.
- Proof that the sphere is a regular surface by writing it as the union of six graphs over the coordinate planes. What happens at the edges of the coordinate charts?
- Another example where the coordinates are differentiable at q but $D\sigma_q$ is non-injective: the sphere in polar coordinates.
- Example: graphs are regular surfaces.
- Example: inverse images of a regular values are regular surfaces, again is based on the ImpFT.
  - Here we have $F(p) = 0$ and $DF_p \neq 0$ meaning $\exists\, i$ so that $\frac{\partial F(p)}{\partial x^i} \neq 0$.
  - W.l.o.g. $i = n$ so we get from the ImpFT the local solution $x^n = g(x^1, \dots, x^{n-1},)$ so that $F(x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})) = 0$.
  - Now $F^{-1}(0)$ near p projects down onto an open set U in the $(x^1, \dots, x^{n-1})$-plane and is equal to the graph $\{(x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})) : (x^1, \dots, x^{n-1}) \in U\}$. Thus it's a surface!

Geometry versus topology.

- Explain this dichotomy.
- Euler characteristic.

The tangent space of a surface.

- Curves in a surface. The coordinate curves. Tangent vectors to a surface.
- Let $\sigma : U \subseteq \mathbb{R}^2 \to V \cap S \subseteq \mathbb{R}^3$ be a parametrization of a subset of a surface S and let $p \in S$ such that $p = \sigma(u)$ for some $u \in U$. The tangent plane T_pS defined as $Image(D\sigma_u) \subseteq T_{\sigma(u)}\mathbb{R}^3$.
- The previous definition depends on the parametrization σ. What if we change parametrizations? Do we get the same tangent space? Yes we do! Do change-of-parameters calculation.
- This is an example of a general principle of differential geometry: to define a *geometric concept* such as the tangent plane rigorously, we can use a parametrization; but then we must show independence of the particular parametrization chosen.
- Basis for the tangent space. This is NOT a geometric concept.
- Tangent space of a graph and of a level set.