CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 9 Supplement

1. The First Variation of Area

The mean curvature of a surface S is the *gradient of surface area.* I.e. the surface area of S decreases fastest when it is deformed by H in the unit normal direction. To see this: let $\phi : \mathcal{U} \to \mathbb{R}^3$ parametrize S and let $f : \mathcal{U} \to \mathbb{R}$ be a function. Then $\phi_\varepsilon := \phi + \varepsilon f \cdot N$ parametrizes a small deformation of S when ε is sufficiently small. Here N is the unit normal vector field — note that a large class of nearby surfaces can be parametrized in this way (think about this!). Denote the deformed surface by S_ε.

We'd like to differentiate the quantity $(Area)(S_\varepsilon)$ and see what comes out. Let $g_\varepsilon(u) := [D\phi_\varepsilon]_u^\top [D\phi_\varepsilon]_u$ and $g := g_0$. Then we can express the area of $\phi_\varepsilon(\mathcal{U})$ as

$$\mathrm{Area}(\phi_\varepsilon(\mathcal{U})) = \int_{\mathcal{U}} \sqrt{\det(g_\varepsilon(u))}du\,.$$

To differentiate this, we'll need a formula for the derivative of a determinant. The formula we'll use is a "standard" result but that you may not have seen yet. Suppose A_ε is a differentiable family of invertible matrices. Then

$$\frac{d}{d\varepsilon}\det(A_\varepsilon) = \det(A_\varepsilon)\mathrm{Tr}\left(A_\varepsilon^{-1}\frac{dA_\varepsilon}{d\varepsilon}\right).$$

A way to re-derive this formula in case you forget is to assume A_ε is diagonal and apply the product rule to the derminant of A_ε, which is the product of the eigenvalues of A_ε. (The actual proof is close to this.) Now we have

$$\begin{aligned}\frac{d}{d\varepsilon}\mathrm{Area}(\phi_\varepsilon(\mathcal{U}))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{\mathcal{U}}\sqrt{\det(g_\varepsilon(u))}du\Big|_{\varepsilon=0} \\ &= \frac{1}{2}\int_{\mathcal{U}}\mathrm{Tr}\Big(g^{-1}\frac{dg_\varepsilon(u)}{d\varepsilon}\Big|_{\varepsilon=0}\Big)\sqrt{\det(g(u))}\,du\,.\end{aligned}$$

To finish, we need to differentiate g_ε. Recall that $[g_\varepsilon]_{ij} = \langle E_i(\varepsilon), E_j(\varepsilon)\rangle$ where

$$E_i(\varepsilon) = \frac{\partial\phi_\varepsilon}{\partial u^i} = E_i(0) + \varepsilon\frac{\partial f}{\partial u^i}N + \varepsilon f(u)\frac{\partial N}{\partial u^i}$$

are the coordinate vector fields. Hence

$$[g_\varepsilon]_{ij} = [g_0]_{ij} + \varepsilon f\left(\left\langle E_i(0), \frac{\partial N}{\partial u^j}\right\rangle + \left\langle\frac{\partial N}{\partial u^i}, E_j(0)\right\rangle\right) + \mathcal{O}(\varepsilon^2) = [g_0]_{ij} + 2\varepsilon f A_{ij} + \mathcal{O}(\varepsilon^2)$$

by the self-adjointness of the second fundamental form A of S at u in the parameter domain $\mathcal{U}$. We've kept track of g_ε up to first order in ε only because we'll eventually take the derivative and then set $\varepsilon = 0$. In fact, we get

$$\frac{d}{d\varepsilon}\mathrm{Area}(\phi_\varepsilon(\mathcal{U}))\Big|_{\varepsilon=0} = \int_{\mathcal{U}} f(u)\mathrm{Tr}\big([g(u)]^{-1}A(u)\big)\sqrt{\det(g(u))}\,du = \int_{\mathcal{U}} f(u)H(u)\sqrt{\det(g(u))}\,.$$

There's a small technicality at work here: the mean curvature is actually the sum of the eigenvalues of the shape operator T which we defined indirectly by the formula $\langle T(E_i), E_j\rangle = A_{ij}$. As mentioned in a previous document, the eigenvalues of T are not equal to those of A unless $\{E_i\}$ is an orthonormal basis. But we can show that the eigenvalues of T are equal to those of $g^{-1}A$. The matrix g^{-1} corrects for the non-orthonormality of the basis.