# CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 7 Supplement

## 1. The Second Fundamental Form and the Shape Operator

We defined the differential of the Gauss map of a surface $S$ at $p \in S$ as the linear mapping $Dn_p : T_pS \to T_pS$. Another name for this is the *shape operator* (actually, $-Dn_p$ is the shape operator). Associated to the shape operator is the self-adjoint quadratic form $A_p(V, W) := -\langle Dn_p(V), W\rangle$ called the *second fundamental form*. A possible point of confusion from lecture today concerns the principal curvatures and directions — what matrix are they the eigenvalues and eigenvectors of?

Here is an explanation. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with associated quadratic form $Q(V, W) := \langle M(V), W\rangle$. Let's assume that $M$ is symmetric and so $Q$ is self-adjoint. Define

$$k_{min} := \min_{\|V\|=1} Q(V, V) \qquad \text{and} \qquad k_{max} := \max_{\|V\|=1} Q(V, V)\,.$$

Then both $k_{min}$ and $k_{max}$ are eigenvalues of $M$. Let $V_{min}$ and $V_{max}$ be the associated eigenvectors. Then $V_{min} \perp V_{max}$ and can be chosen of unit length. This holds true even when $k_{min} = k_{max}$; now the eigenvalues are degenerate and any orthonormal vectors will do! Next, it is the case that

$$\mathrm{Tr}(M) = k_{min} + k_{max} \qquad \text{and} \qquad \det(M) = k_{min} \cdot k_{max}\,.$$

To actually compute these quantities, we need to choose a basis. Note that the matrix entries of $M$ with respect to a basis $E_1, E_2$ are defined as the coefficients in the expansion $M(E_i) := \sum_j M_{ij}E_j$. Therefore the matrix entries satisfy $M_{ij} = \langle M(E_i), E_j\rangle = Q(E_i, E_j) = Q_{ij}$ if and only if the basis is orthonormal. In this case

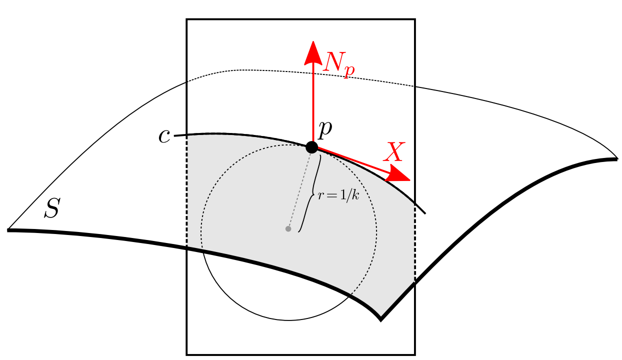
$$k_{min} + k_{max} = Q_{11} + Q_{22} \qquad \text{and} \qquad k_{min} \cdot k_{max} = Q_{11}Q_{22} - Q_{12}^2\,.$$

Otherwise, let $g = \begin{pmatrix} \|E_1\|^2 & \langle E_1, E_2\rangle \\ \langle E_1, E_2\rangle & \|E_2\|^2 \end{pmatrix}$ and then one can show that

$$k_{min} + k_{max} = \sum_{ij} [g^{-1}]_{ij} Q_{ij} \qquad \text{and} \qquad k_{min} \cdot k_{max} = \frac{Q_{11}Q_{22} - Q_{12}^2}{\det(g)}\,.$$

## 2. Local "Shape" of a Surface

**A nicer picture.** The picture I drew on the board for explaining the relation between the second fundamental form $A_p$ of a surface $S$ at $p$ and the geodesic curvature of curves on $S$ passing through $p$ wasn't very good. Here is a better picture.




I'm drawing $S$ together with a vector $X \in T_pS$ and a curve passing through $p$ in direction $X$. I've obtained $c$ by intersecting $S$ with the plane passing through $p$ spanned by $X$ and the normal vector $N_p$. I've also drawn a circle in this plane that makes *second order contact* with the curve $c$ at $p$. This circle has radius equal to one over the geodesic curvature $k_c(0)$; and by our formula, we also know that $k_c(0) = A_p(X, X)$.

**Classification of surface points by their curvature.** From your homework assignment, we know that every surface is locally the graph of a function over its tangent plane. So without loss of generality, we can analyze the second fundamental form in the following setting. Let $S := \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\}$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with $f(0, 0) = 0$ and $\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0$. You also know from your homework assignment that the tangent vectors there are $E_1 = (1, 0, 0)^\top$ and $E_2 = (0, 1, 0)^\top$ while the second fundamental form of $S$ there is

$$[A_0]_{ij} = -\frac{\partial^2 f(0,0)}{\partial x^i \partial x^j}$$

Moreover, we know from Taylor's theorem that

$$f(x, y) = \frac{1}{2}(x, y)D^2 f(0, 0)(x, y)^\top + \mathcal{O}(\|(x, y)\|^3) = -\frac{1}{2}A_0\big((x, y)^\top, (x, y)^\top\big) + \mathcal{O}(\|(x, y)\|^3)\,.$$

Hence if $A_0$ is non-zero as a quadratic form, then $A_0$ characterizes the local shape of $S$ near the origin. That is, we can classify the origin as one of several different types:

- The origin is an *elliptic point* if either $k_{min} > 0$ and $k_{max} > 0$, or $k_{min} < 0$ and $k_{max} < 0$.
- It is a *hyperbolic point* if $k_{min} < 0$ and $k_{max} > 0$
- It is a *parabolic point* if one of $k_{min} = 0$ or $k_{max} = 0$.
- It is a *planar point* if $k_{min} = k_{max} = 0$.
- It is an *umbilic point* if $k_{min} = k_{max}$. The key feature here is that the principal directions are not uniquely defined.

We can see examples of each kind of point by choosing different functions $f : \mathbb{R}^2 \to \mathbb{R}$. For instance, we can get examples of the first three kinds (and the last kind) by choosing $f(x, y) = k_{min}x^2 + k_{max}y^2$ which is either a paraboloid (up or down) or a hyperboloid or a degenerate quadratic form depending on the signs of the principal curvatures and whether one of them is zero or not. We get an example of the fourth kind by choosing $f(x, y) = ax + by$ — in other words $S$ is a plane.

## 3. Interpretations of the Mean and Gauss Curvatures

We'll need this material for Wednesday's lecture. The results will be stated here — and we'll discuss the proof of these results briefly next Monday.

**Mean curvature as first variation of area.** Let $S$ be an orientable surface and consider a *deformation* of $S$ constructed in the following way. Choose a function $f : S \to \mathbb{R}$ and a small number $\varepsilon > 0$ and displace each $p \in S$ by an amount $\varepsilon f(p)$ in the normal direction at $p$. In other words $p_{displaced} := p + \varepsilon f(p)N_p$. The new surface is $S_\varepsilon := \{p_{displaced} : p \in S\}$.

Now as $S$ deforms into $S_\varepsilon$, its surface area changes. We will see that

$$\frac{d}{d\varepsilon}\mathrm{Area}(S_\varepsilon)\bigg|_{\varepsilon=0} = -\int_S f(p)H(p)dArea(p)\,.$$

In other words, the first order change in the area is given by integration against the mean curvature. This also means that if $f(p) = H(p)$ then the surface area *decreases the fastest*. In other words, we can interpret the mean curvature as the *gradient of the surface area functional*.

**Gauss curvature in terms of the Gauss map.** This time we keep the surface $S$ fixed and consider small balls about a point $p \in S$. where $K(p) \neq 0$. Let $\varepsilon > 0$ be such that $K$ does not change sign in $B_\varepsilon(p)$. Then if $n$ denotes the Gauss map, we will show that

$$K(p) = \lim_{\varepsilon \to 0} \frac{Area\big(n(B_\varepsilon(p))\big)}{Area(B_\varepsilon(p))}$$