CS 468

Differential Geometry for Computer Science

Lecture 7 — Extrinsic Curvature

Outline

- Normal vectors.
- Surface integrals and surface area.
- The Gauss map.
- The second fundamental form.
- Interpretation — extrinsic curvature.

The Unit Normal Vector of a Surface

- The unit normal vector of a surface. Is this geometric?
  - → Normal line is geometric. Normal direction may not be.
  - → Non-orientable surfaces.

- Unit normal vector of a parametrized surface and a graph:

$$If \quad T_pS = span\{E_1, E_2\} \quad \text{then} \quad N := \frac{E_1 \times E_2}{\|E_1 \times E_2\|}$$

- Unit normal vector of level set $S := F^{-1}(v)$ at regular value v. Let c be a curve $\subseteq S$ with $c(0) = p$ and $\dot{c}(0) = X \in T_pS$.

$$\Rightarrow \qquad v = F(c(t)) \quad \forall\, t$$

$$\Rightarrow \qquad 0 = \frac{d}{dt} F(c(t))\Big|_{t=0} = \langle [DF_p]^\top, X \rangle$$

$$\Rightarrow \qquad N := \frac{[DF_p]^\top}{\|DF_p\|} \perp T_pS$$

Surface Area

- Area of infinitesimal coordinate rectangle.
  - → Let $\phi : \mathcal{U} \to \mathbb{R}^3$ be a parametrization of S.
  - → Let $E_i := D\phi_u \cdot [0, \ldots, 1, \ldots, 0]^\top$ span $T_{\phi(u)}S$.
  - → Area of rectangle $E_1 \wedge E_2$ is $\|E_1 \times E_2\| = |\det(D\phi_u^\top D\phi_u)|^{1/2}$.
- The Riemann sum that yields a surface integral.

  Let $f : S \to \mathbb{R}$ be an integrable function.

$$\Rightarrow \qquad \int_S f \, d\text{Area} := \lim \sum_i f(\phi(u_i)) \sqrt{\det(D\phi_{u_i}^\top D\phi_{u_i})}$$

- where the Riemannian area form is:

$$d\text{Area}(u) := \sqrt{\det(D\phi_u^\top D\phi_u)}\, du^1 \, du^2$$

- Independence of parametrization.

The Gauss Map

Let S be an orientable surface with unit normal vector N_p at $p \in S$.

- The Gauss map of S is the mapping $n : S \to \mathbb{S}^2$ given by
$$n(p) := N_p$$
- We view N_p as a vector in $\mathbb{R}^3$ of length one $\Rightarrow$ a point in $\mathbb{S}^2$.
- The Gauss map of a differentiable surface is itself differentiable.
- Its differential is $Dn_p : T_pS \to T_{N_p}\mathbb{S}^2$.
- Since T_pS and $T_{n(p)}\mathbb{S}^2$ are parallel planes (they're both perpendicular to N_p), we can re-define $Dn_p : T_pS \to T_pS$.

The Second Fundamental Form

Defn: The second fundamental form of S at p is the bilinear form

$$A_p : T_pS \times T_pS \to \mathbb{R}$$
$$A_p(V, W) := -\langle Dn_p(V), W\rangle$$

It measures the projection onto W of the rate of change of N_p in the V-direction at every point $p \in S$.

Proposition: A_p is self-adjoint.

Proof: Work with the components $[A_p]_{ij} := \left\langle \frac{\partial N}{\partial u^i}, \frac{\partial \phi}{\partial u^j} \right\rangle$. The key is the symmetry of mixed partial derivatives!

Extrinsic Curvature

- Let c be a curve in S with $c(0) = p$.
- Let $\vec{k}_c(0)$ be the geodesic curvature vector of c at zero. Then

$$\langle \vec{k}_c(0), N_p \rangle = A_p(\dot{c}(0), \dot{c}(0))$$

- Note: depends only on the geometry of S at p.
- Let V vary over all unit vectors in T_pS. Then $A_p(V, V)$ takes on a minimum value k_{min} and a maximum value k_{max}.
  - → Eigenvalues of A_p — the *principal curvatures* of S at p.
  - → Corresponding eigenvectors are the *principal directions* of S at p.
  - → Note that the principal directions are orthogonal.
- Mean curvature $H := k_{min} + k_{max}$ $(= \mathrm{Tr}(A_p)$ w.r.t. ONB$)$.
- Gauss curvature $K := k_{min} \cdot k_{max}$ $(= \det(A_p)$ w.r.t. ONB$)$.

Local Shape of a Surface

Example: Second fundamental form of a graph. What can we see?

- *Elliptic*, *hyperbolic*, *parabolic*, *planar* or *umbilic* points.
- Local characterization of the surface S at p depending on type.
  - $\rightarrow$ Proof based on Taylor series expansion.

Interpretation of the Mean Curvature

The mean curvature is the gradient of the area functional.

- I.e. area decreases fastest in the $H\vec{n}$ direction.

The calculation:

- Let $\phi : \mathcal{U} \to \mathbb{R}^3$ parametrize S and let $f : \mathcal{U} \to \mathbb{R}$ be a function.
- Let $\phi_\varepsilon(u) := \phi(u) + \varepsilon f(u) N_u$ parametrize a deformation of S.

Now let $g_\varepsilon(u) := [D\phi_\varepsilon]_u^\top [D\phi_\varepsilon]_u$ and $g := g_0$. Then

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathrm{Area}(\phi_\varepsilon(\mathcal{U}))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_{\mathcal{U}} \sqrt{\det(g_\varepsilon(u))} du \Big|_{\varepsilon=0} \\
&= \int_{\mathcal{U}} \mathrm{Tr}\left(g^{-1} \frac{dg_\varepsilon(u)}{d\varepsilon}\Big|_{\varepsilon=0} \right) \sqrt{\det(g(u))}\, du \\
&= -2 \int_{\mathcal{U}} H(u) f(u) \sqrt{\det(g(u))}\, du
\end{aligned}
$$

Interpretation of the Gauss Curvature

Two results. Let n be the Gauss map.

- **Proposition:** $K(p) > 0$ iff n locally preserves orientation; and $K(p) < 0$ iff n locally reverses orientation.
- **Proposition:** Let $p \in S$ be such that $K(p) \neq 0$ and let $\varepsilon > 0$ be such that K does not change sign in $B_\varepsilon(p)$. Then we have

$$K(p) = \lim_{\varepsilon \to 0} \frac{Area\big(n(B_\varepsilon(p))\big)}{Area(B_\varepsilon(p))}$$