CS 468 (Spring 2013) — Discrete Differential Geometry

Lectures 5: Surface Geometry

Level sets.

- Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$ be a number. The level set of F with value c is the set of points
$$F^{-1}(c) := \{p \in \mathbb{R}^3 : F(p) = c\}$$
- So to find a level set, you must solve the equation $F(p) = c$ for $p = (x, y, z)$.
- Note: if there are no solutions, then $F^{-1}(c) = \varnothing$ (the empty set).

Level sets as surfaces.

- The big question: what is the geometric nature of a level set?
- Our intuition says the a level set is a surface because a level set consists of the solution of "one scalar equation in three unknowns."
- The reasoning is: by solving the equations you should be able to express one of the unknowns as a function of the other two. In other words, you can write $z = g(x, y)$ for some function g, and $F(x, y, g(x, y)) = c$. Now the solution set looks like
$$\{(x, y, g(x, y)) : x, y2 \in U \subseteq \mathbb{R}^2\}$$
In other words, the solution set is a graph, which is a surface as we saw in class.

Exceptions.

- There are exceptions to the nice intuitive picture described above.
- For example, consider the function $F(x, y, z) := x^2 + y^2 + z^2$. The level set of $c > 0$ is a sphere of radius $\sqrt{c}$ — which is a surface. The level set of $c < 0$ is the empty set. The level set of $c = 0$ consists of the point $(0, 0, 0)$ only. In other words, $F^{-1}(0) = \{(0, 0, 0)\}$. This is not a surface.
- There are other examples where $F^{-1}(c)$ is not a surface. For instance, the level set of zero of the function $F(x, y, z) = x^2 + y^2$ is the z-axis, which is a line and not a surface. (Other level sets with $c > 0$ are cylinders and with $c < 0$ are the empty set.)
- Even weirder things can happen. For instance, the level set of zero of the function $F(x, y, z) := xy$ is the *union* of the (y, z)-plane and the (x, z)-plane which is not a surface in the neighbourhood of the z-axis. (Draw this object!)
- Much, much weirder things can happen.

Regular values.

- We would like to characterize when a level set is a surface. We will need the concept of a *regular value.*
- Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. A number $c \in R$ is a regular value for F if the derivative matrix of F (which is a 1×3 matrix in this case) does not vanish anywhere on the level set $F^{-1}(c)$.
- I.e. c is a regular value for F if $DF_p = (\frac{\partial F(p)}{\partial x}, \frac{\partial F(p)}{\partial y}, \frac{\partial F(p)}{\partial z}) \neq (0, 0, 0)$ for all $p \in F^{-1}(0)$.

The inverse image of a regular value is a surface.

- Suppose c is a regular value for F and let $p \in F^{-1}(c)$.
- Without loss of generality, we can assume that $\frac{\partial F(p)}{\partial z} \neq 0$.
- We now invoke the Implicit Function Theorem.
- Write $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$. Now the matrix $D_2 F_p$ appearing in this theorem is simply the number $\frac{\partial F(p)}{\partial z}$. So the invertibility of $D_2 F_p$ is equivalent to $\frac{\partial F(p)}{\partial z} \neq 0$.
- The Implicit Function Theorem now gives us a local solution $z = g(x, y)$ where $g : U \to \mathbb{R}$ is a smooth function defined in a neighbourhood of p.
- Now $F^{-1}(0)$ near p can be parametrized with the help of g. That is, we can write $F^{-1}(0)$ near p as $\{(x, y, g(x, y)) : (x, y) \in U\}$.
- In other words, $F^{-1}(0)$ near p is the graph of g. This is a regular surface!

A nice formula.

- We can relate the derivatives of g to the derivatives of F using the chain rule.
- We have $F(x, y, g(x, y)) = c$ so for instance

$$\begin{aligned} 0 &= \frac{\partial F(x, y, g(x, y))}{\partial x} \\ &= \frac{\partial F}{\partial x}(x, y, g(x, y)) + \frac{\partial F}{\partial z}(x, y, g(x, y)) \cdot \frac{\partial g}{\partial x} \end{aligned}$$

- By isolating $\frac{\partial g}{\partial x}$ we obtain the formula

$$\frac{\partial g}{\partial x} = -\frac{\frac{\partial F}{\partial x}(x, y, g(x, y))}{\frac{\partial F}{\partial z}(x, y, g(x, y))}$$

  which is a sensible mathematical expression so long as $\frac{\partial F}{\partial z} \neq 0$ which is certainly true sufficiently close to p.
- A similar formula holds for $\frac{\partial g}{\partial y}$.