CS 468

Differential Geometry for Computer Science

Lecture 2 — Curves

Definition of a curve

- A parametrized differentiable curve is a differentiable map $\gamma : I \to \mathbb{R}^n$ where $I = (a, b)$ is an interval in $\mathbb{R}$.
- The parameter domain I.
- The image or *trace* of γ.
- The *component functions* of γ.

Velocity and Acceleration

- Instantaneous velocity.
- Instantaneous acceleration.
- Constant speed curves and constant velocity curves.
- Singular points.

Examples

- Lines in space.
- Circle in $\mathbb{R}^2$.
- Helix in $\mathbb{R}^3$.
- Self-intersection — embedded vs. immersed curves.
- Curve with a kink, curve with a cusp — smooth but singular, and non-smooth parametrizations thereof.

Change of parameter

- Definition of reparametrization.
- The trace remains unchanged.
- Effect on velocity and acceleration.

Arc-length

- Arc-length is the limit of a sequence of discrete approximations.
- Derivation: let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth curve and partition $I = [t_0, t_1] \cup \cdots \cup [t_{n-1}, t_n]$ with $t_0 = a$ and $t_n = b$. Now

$$
\begin{aligned}
length(\gamma([a,b])) &\approx \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
&= \sum_{i=1}^{n} \|\dot{\gamma}(t_i^*)\Delta t_i\| \\
&= \sum_{i=1}^{n} \|\dot{\gamma}(t_i^*)\|\Delta t_i \\
&\xrightarrow{n\to\infty} \int_a^b \|\dot{\gamma}(t)\| dt
\end{aligned}
$$

Parameter independence of arc-length

- Let $\phi : [a,b] \to [a,b]$ be a diffeomorophism with $\phi(a) = a$ and $\phi(b) = b$. Let $\tilde{\gamma}(s) := \gamma(\phi(s))$. Then

$$
\begin{aligned}
length(\tilde{\gamma}([a,b])) &= \int_a^b \left\| \frac{d\gamma \circ \phi(s)}{ds} \right\| ds \\
&= \int_a^b |\phi'(s)| \left\| \frac{d\gamma}{dt} \circ \phi(s) \right\| ds \\
&= \int_a^b |\phi' \circ \phi^{-1}(t)| \left\| \frac{d\gamma(t)}{dt} \right\| \frac{dt}{|\phi' \circ \phi^{-1}(t)|} \\
&= \int_a^b \left\| \frac{d\gamma(t)}{dt} \right\| dt \\
&= length(\gamma([a,b]))
\end{aligned}
$$

Example calculations

- Mostly no closed form for arc-lengths.
- First example: logarithmic spiral $\gamma(t) = (e^t \cos(t), e^t \sin(t))$.
- Second example: $\gamma(t)$ such that $\|\dot{\gamma}\| = const$.
- Parametrization by arc-length.

Arc-length re-parametrization

- We can *re-parametrize* any curve so that it is parametrized by arc-length. (Useful theoretically but hard to put into practice.)
- Let $\gamma : I \to \mathbb{R}$ be a smooth curve and define the function

$$\ell : I \to [0, \mathit{length}(\gamma(I))]$$
$$\ell(t) := \int_0^t \|\dot{\gamma}(x)\| dx$$

- Invertibility of ℓ when γ is non-singular.
- Define a new parameter s that satisfies $s = \ell(t)$. Define the *re-parametrized version* of γ, namely $\tilde{\gamma}(s) = \gamma(\ell^{-1}(s))$.
- This re-parametrized version is parametrized by arc-length.
- Example: the logarithmic spiral.

Curvature

- Definition of the geodesic curvature vector in an arbitrary parametrization — the normal component of the acceleration vector, normalized by the squared length of the tangent vector.

$$\vec{k}_c := \frac{[\ddot{c}]^\perp}{\|\dot{c}\|^2}$$

- Definition of the geodesic curvature $k_c := \|\vec{k}_c\|$.
- In the arc-length parametrization we have $\vec{k}_c = [\ddot{c}]^\perp$.
- Examples.

The Frenet frame

- Let $\gamma :\to \mathbb{R}^3$ be a curve, w.l.o.g parametrized by arc-length.
- We will find a choice of "moving axes best adapted to the geometry of γ.
- Let $T(s) := \dot{\gamma}(s)$.
- A point of non-zero curvature allows us to define a distinguished normal vector $N(s) := \dot{T}(s)/\|\dot{T}(s)\|$.
- The *osculating plane* at $\gamma(s)$ is spanned by $T(s)$, $N(s)$.
- The *binormal vector* is $B(s) := T(s) \times N(s)$.
- The *Frenet frame* for γ is $\{T(s), N(s), B(s)\}$ and is defined at each point $\gamma(s)$ where $k_\gamma(s) \neq 0$.

The Frenet formulas

- The Frenet formulas explain the variation in the Frenet frame along γ.

$$\dot{T}(s) = k_\gamma(s) N(s)$$

$$\begin{aligned}\dot{N}(s) &= \langle \dot{N}(s), T(s)\rangle T(s) + \langle \dot{N}(s), N(s)\rangle N(s) + \langle \dot{N}(s), B(s)\rangle B(s)\\ &= -k_\gamma(s) T(s) + \langle \dot{N}(s), B(s)\rangle B(s)\\ &= -k_\gamma(s) T(s) - \tau_\gamma(s) B(s)\end{aligned}$$

$$\begin{aligned}\dot{B}(s) &= \langle \dot{B}(s), T(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s) + \langle \dot{B}(s), B(s)\rangle B(s)\\ &= -\langle B(s), \dot{T}(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s)\\ &= -k_\gamma(s)\langle B(s), N(s)\rangle T(s) - \langle N(s), \dot{B}(s)\rangle B(s)\\ &= \tau_\gamma(s) B(s)\end{aligned}$$

- Here we have introduced the *torsion* $\tau_\gamma(s) := -\langle \dot{N}(s), B(s)\rangle$.

A local theorem

- Locally, k and $\dot{k}$ determine the amount of turning in the osculating plane.
- And τ and k determine the amount of lifting out of the osculating plane into its normal direction.

A global theorem

- The Fundamental Theorem of Curves.

Suppose we are give differentiable functions $k : I \to \mathbb{R}$ with $k > 0$, and $\tau : \to \mathbb{R}$.

Then there exists a regular curve $\gamma : I \to \mathbb{R}^3$ such that s is the arc-length, $k(s)$ is the geodesic curvature, and $\tau(s)$ is the torsion.

Any other curve satisfying the same conditions differs from γ by a rigid motion.