CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 2: Curves

Definition of a curve.

- Definition. A parametrized differentiable curve in $\mathbb{R}^n$ is a differentiable map $\gamma : I \to \mathbb{R}^n$ where $I = (a, b)$ is an open interval in $\mathbb{R}$. Note: I can be a closed interval — now we have a curve with boundary points.
- Notation. Such a map has component functions $\gamma(t) := (\gamma_1(t), \ldots, \gamma_n(t))$. Each $\gamma_i : I \to \mathbb{R}$ is a differentiable function.
- The domain I is the space where the parameter t lives.
- The image of γ is the set of points $\{\gamma(t) : t \in I\} \subseteq \mathbb{R}^n$. It is a geometric thing called the *trace* of the curve. We interpret $\gamma(t)$ as the location of a particle in space at the instant of time t; and we interpret the trace of the curve as the path traced out by the particle as t varies in I.
- Distinction between this kind of curve and a 1-D manifold.

Velocity and Acceleration.

- Instantaneous velocity of the particle at time t is $\dot{\gamma}(t) = (\dot{\gamma}_1(t), \ldots, \dot{\gamma}_n(t))$.
- Instantaneous acceleration of the particle at time t is $\ddot{\gamma}(t) = (\ddot{\gamma}_1(t), \ldots, \ddot{\gamma}_n(t))$.
- Constant speed curves; acceleration is normal to the velocity. Constant velocity curves are straight lines.
- Singular points where $\dot{\gamma} = 0$. The parametrized map can still be differentiable but the trace may not be smooth. For example:

$$\gamma(t) := \begin{cases} (e^{-1/t^2}, 0) & t > 0 \\ 0 & t = 0 \\ (0, e^{-1/t^2}) & t < 0 \end{cases}$$

Examples.

- Lines in space: $\gamma(t) = x_0 + tv$ is the line passing through x_0 in direction v.
- Circle in $\mathbb{R}^2$, helix in $\mathbb{R}^3$.
- Curve in which the trace intersects itself
- Curve with a kink, curve with a cusp — smooth (with singular point) and non-smooth parametrizations thereof (e.g. $\gamma(t) = (t^3, t^2)$ or $\bar{\gamma}(t) = (t, t^{2/3})$).
- An exotic example. E.g. Cycloid — the motion of a point on the rim of a wheel of radius R as the wheel rolls without slipping along the x-axis. (This is derived as follows. Let θ be the angle through which the wheel has rolled. Then the distance the point of contact with the ground has moved is equal to $R\theta$. Hence the position of the centre of the wheel has moved to $(R\theta, R)$. And the point on the edge of the wheel, originally touching the ground at $\theta = 0$ has rotated through a clockwise angle of θ measured relative to the centre of the wheel. In other words, this point is located at

$$\gamma(\theta) := (R\theta, R) + (R\cos(-\pi/2 - \theta), R\sin(-\pi/2 - \theta)) = (R\theta, R) - (R\sin(\theta), R\cos(\theta))\,.$$

Change of parameter.

- Definition of reparametrization: a bijective map $\phi : J \to I$ gives you a new curve $\tilde{\gamma} : J \to \mathbb{R}^n$ defined by $\tilde{\gamma}(s) = \gamma(\phi(s))$. The formula $t = \phi(s)$ is a *change of parameter*.
- Note that a smooth mapping ϕ between intervals is a bijection if and only if ϕ' never vanishes.
- The trace remains unchanged.
- Effect on velocity and acceleration:

$$\begin{aligned} \frac{d\tilde{\gamma}(s)}{ds} &= \frac{d\gamma(\phi(s))}{ds} = \frac{d\gamma}{dt} \circ \phi(s) \frac{d\phi(s)}{ds} \qquad \text{Note length changes} \\ \frac{d^2\tilde{\gamma}(s)}{ds^2} &= \frac{d}{ds}\left(\frac{d\gamma}{dt} \circ \phi(s) \frac{d\phi(s)}{ds}\right) \\ &= \frac{d^2\gamma}{dt^2} \circ \phi(s) \left(\frac{d\phi(s)}{ds}\right)^2 + \frac{d\gamma}{dt} \circ \phi(s) \frac{d^2\phi(s)}{ds^2} \end{aligned}$$

Arc length.

- Discrete approximation of the length of a differentiable curve by means of line segments; limit as segment length $\to 0$ yields the arc length integral.
- Derivation: let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth curve and partition $I = [t_0, t_1] \cup \cdots \cup [t_{n-1}, t_n]$ with $t_0 = a$ and $t_n = b$. Suppose $\gamma(t) = (x(t), y(t), z(t))$. Now compute

$$\begin{aligned} \mathit{length}(\gamma([a, b])) &\approx \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &= \sum_{i=1}^{n} \left((x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2\right)^{1/2} \\ &= \sum_{i=1}^{n} \left((\dot{x}(t_i^*)\Delta t_i)^2 + (\dot{y}(t_i^*)\Delta t_i)^2 + (\dot{z}(t_i^*)\Delta t_i)^2\right)^{1/2} \qquad \text{Mean value theorem; } t_i^* \in [t_{i-1}, t_i] \text{ and } \Delta_i := |t_i - t_{i-1}| \\ &= \sum_{i=1}^{n} \left((\dot{x}(t_i^*))^2 + (\dot{y}(t_i^*))^2 + (\dot{z}(t_i^*))^2\right)^{1/2} \Delta t_i \\ &\xrightarrow{n \to \infty} \int_a^b \left((\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2\right)^{1/2} dt \\ &= \int_a^b \|\dot{\gamma}(t)\| dt \end{aligned}$$

- Parameter independence. Let $\phi : [a, b] \to [a, b]$ be a diffeomorophism with $\phi(a) = a$ and $\phi(b) = b$. Let $\tilde{\gamma}(s) := \gamma(\phi(s))$. Then

$$\begin{aligned} \mathit{length}(\tilde{\gamma}([a, b])) &= \int_a^b \left\|\frac{d\gamma \circ \phi(s)}{ds}\right\| ds \\ &= \int_a^b |\phi'(s)| \left\|\frac{d\gamma}{dt} \circ \phi(s)\right\| ds \qquad \text{Let } t = \phi(s) \text{ so } dt = \phi'(s)ds \text{ and thus } ds = (\phi'(s))^{-1}dt = (\phi' \circ \phi^{-1}(t))^{-1}dt \\ &= \int_a^b |\phi' \circ \phi^{-1}(t)| \left\|\frac{d\gamma(t)}{dt}\right\| \frac{dt}{|\phi' \circ \phi^{-1}(t)|} \\ &= \int_a^b \left\|\frac{d\gamma(t)}{dt}\right\| dt \\ &= \mathit{length}(\gamma([a, b])) \end{aligned}$$

- Example calculations — mostly no closed form for arc lengths.
  - First example: $\gamma(t) = (e^t \cos(t), e^t \sin(t))$. Then $\dot\gamma(t) = e^t(\cos(t), \sin(t)) + e^t(-\sin(t), \cos(t))$ and $\|\dot\gamma(t)\| = e^t \|(\cos(t), \sin(t)) + (-\sin(t), \cos(t))\| = \sqrt{2}e^t$. Thus

$$\mathit{length}(\gamma([0,T])) = \int_0^T \|\dot\gamma(t)\| dt = \sqrt{2}\int_0^T e^t dt = \sqrt{2}(e^T - 1)$$

  - Second example: $\gamma(t)$ such that $\|\dot\gamma\| = \mathit{const}$. Then

$$\mathit{length}(\gamma([T_0,T])) = \int_{T_0}^T \|\dot\gamma(t)\| dt = C(T - T_0)$$

    Thus $L = C(T - T_0)$ and T is *almost* the arc-length parameter itself. If $C = 1$ we say that γ is *parametrized by arc-length.*

- The arc length re-parametrization — proof that it has constant velocity. Let $\gamma : I \to \mathbb{R}$ be a smooth curve and define the function $\ell : I \to [0, \mathit{length}(\gamma(I))]$ by $\ell(t) := \int_0^t \|\dot\gamma(x)\| dx$.
  - Note that $\frac{d\ell(t)}{dt} = \|\dot\gamma(t)\|$ so that if γ has no points where $\dot\gamma = 0$ then ℓ is invertible.
  - Define a new parameter s that satisfies $s = \ell(t)$. So now we have $t = \ell^{-1}(s)$ and we can define a *re-parametrized version* of γ, namely $\tilde\gamma(s) = \gamma(\ell^{-1}(s))$.
  - Note that $\|\frac{d}{ds}\tilde\gamma(s)\| = 1$ because

$$\frac{d\tilde\gamma(s)}{ds} = \frac{d\gamma}{dt} \circ \ell^{-1}(s) \frac{d\ell^{-1}(s)}{ds} = \frac{\dot\gamma \circ \ell^{-1}(s)}{\frac{d\ell}{dt} \circ \ell^{-1}(s)} = \frac{\dot\gamma \circ \ell^{-1}(s)}{\|\dot\gamma \circ \ell^{-1}(s)\|}$$

  - Thus $\|\frac{d\tilde\gamma(s)}{ds}\| = 1$ and the re-parametrized version is parametrized by arc length.
  - The arc-length parametrization is very useful theoretically (as we'll see) but difficult to work with in practice because the arc-length can be hard to compute (i.e. finding the function ℓ) and it's inverse can then be very hard to find (i.e. inverting to find ℓ^{-1}).
  - Example: we have $s = \sqrt{2}e^t$ for the logarithmic spiral so $t = \log(s/\sqrt{2})$. Hence the re-parametrized version of the logarithmic spiral is

$$\tilde\gamma(s) = \frac{s}{\sqrt{2}}\big(\cos(\log(s/\sqrt{2})), \sin(\log(s/\sqrt{2}))\big)\,.$$

Curvature.

- Definition of the geodesic curvature vector in an arbitrary parametrization — the normal component of the acceleration vector, normalized by the squared length of the tangent vector.

$$\vec{k}_c := \frac{1}{\|\dot c\|^2}\left(\ddot c - \frac{\langle \ddot c, \dot c\rangle}{\|\dot c\|^2}\dot c\right) = \frac{1}{\|\dot c\|}\left[\frac{d}{dt}\left(\frac{\dot c}{\|\dot c\|}\right)\right]^{\perp}$$

Rate of change of the unit tangent vector perpendicular to the curve

- Definition of the geodesic curvature $k_c := \|\vec{k}_c\|$.
- In the arc length parametrization we have $\vec{k}_c = [\ddot c]^{\perp}$.
- Examples: zero-acceleration curve — straight line; constant-acceleration plane curve — circle.

Frenet frame.

- Let $\gamma :\to \mathbb{R}^3$ be a curve, without loss of generality parametrized by arc-length. We will now find a *canonical framing* of γ, namely a choice of "moving axes" (three linearly independent vectors attached to each point $\gamma(s)$) that is best adapted to its geometry.
- Let $T(s) := \dot{\gamma}(s)$. Then $\|T(s)\| = 1$ for all s since γ is parametrized by arc-length.
- A point of non-zero curvature allows us to define a distinguished normal vector. Recall that we have $0 = \frac{d}{ds}\|\dot{\gamma}(s)\|^2 = 2\langle T(s), \dot{T}(s)\rangle = 2\langle T(s), \vec{k}_\gamma(s)\rangle$. Thus the curvature vector is normal to γ. Since it's not equal to zero, we can divide by its magnitude and obtain a unit normal vector field $N(s) := \dot{T}(s)/\|\dot{T}(s)\|$ along γ. This is our second vector in the moving axis.
- We define the *osculating plane* at $\gamma(s)$ to the plane spanned by $T(s)$ and $N(s)$.
- We now define the *binormal vector*, the third vector in our moving axes, to be $B(s) := T(s) \times N(s)$. This is also a unit vector and is orthogonal to both $T(s)$ and $N(s)$.
- The *Frenet frame* for γ is the set of moving axes $\{T(s), N(s), B(s)\}$ and is defined at each point $\gamma(s)$ where $k_\gamma(s) \neq 0$.
- The Frenet formulas explain the variation in the Frenet frame along γ. That is, we have

$$\dot{T}(s) = k_\gamma(s)N(s)$$

$$\begin{aligned}\dot{N}(s) &= \langle \dot{N}(s), T(s)\rangle T(s) + \langle \dot{N}(s), N(s)\rangle N(s) + \langle \dot{N}(s), B(s)\rangle B(s) \\ &= -k_\gamma(s)T(s) + \langle \dot{N}(s), B(s)\rangle B(s) \\ &= -k_\gamma(s)T(s) - \tau_\gamma(s)B(s)\end{aligned}$$

$$\begin{aligned}\dot{B}(s) &= \langle \dot{B}(s), T(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s) + \langle \dot{B}(s), B(s)\rangle B(s) \\ &= -\langle B(s), \dot{T}(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s) \\ &= -k_\gamma(s)\langle B(s), N(s)\rangle T(s) - \langle B(s), \dot{N}(s)\rangle N(s) \\ &= \tau_\gamma(s)N(s)\end{aligned}$$

- Here we have introduced the *torsion* $\tau_\gamma(s) := -\langle \dot{N}(s), B(s)\rangle$.
- Local Theorem: Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a curve with non-zero curvature. Let $k := k_\gamma(0)$ and $\tau = \tau_\gamma(0)$ and $k' = \dot{k}_\gamma(0)$. Then

$$\begin{aligned}\gamma(s) &\approx \gamma(0) + s\dot{\gamma}(0) + \frac{s^2}{2}\ddot{\gamma}(0) + \frac{s^3}{6}\dddot{\gamma}(0) \\ &= \left(s - \frac{k^2s^3}{6}\right)T(0) + \left(\frac{s^2k}{2} + \frac{s^3k'}{6}\right)N(0) - \frac{k\tau s^3}{6}B(0)\end{aligned}$$

  Thus locally, k and k' determine the amount of turning in the $\{T(0), N(0)\}$-plane, while τ and k determine the amount of lifting out of the $\{T(0), N(0)\}$-plane in the $B(0)$-direction.
- Global Theorem: the Fundamental Theorem of Curves.

  "Given differentiable functions $k : I \to \mathbb{R}$ with $k > 0$, and $\tau :\to \mathbb{R}$, there exists a regular curve $\gamma : I \to \mathbb{R}^3$ such that s is the arc-length, $k(s)$ is the geodesic curvature, and $\tau(s)$ is the torsion. Any other curve satisfying the same conditions differs from γ by a rigid motion."

- A proof of the uniqueness part: differentiate $\frac{1}{2}\|\gamma(s) - \tilde{\gamma}(s)\|^2$. A proof of the existence part: involves solving a system of ODEs.

Bishop frame.

- The Frenet frame has an "existential" problem... I. e. it is not defined when $k_\gamma(s) = 0$. But as a paper from the 1960s asserts: *There is more than one way to frame a curve.*
- Definition. The Bishop frame gives an alternative framing of a curve.
- Variational characterization of the Bishop frame. Bending and twisting energies.
- What's the best example?