CS 468

Differential Geometry for Computer Science

Lecture 19 — Conformal Geometry

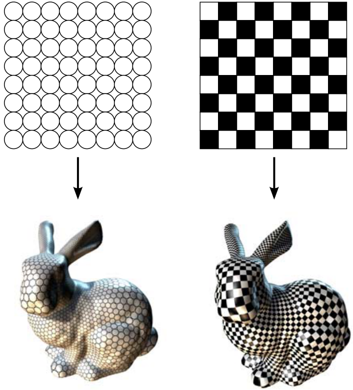
Outline

- Conformal maps
- Complex manifolds
- Conformal parametrization
- Uniformization Theorem
- Conformal equivalence

Conformal Maps

Idea: Shapes are rarely isometric. Is there a weaker condition that is more common?

Conformal Maps



[Gu 2008]

- Let S_1, S_2 be surfaces with metrics g_1, g_2.
- A conformal map preserves angles but can change lengths.
- I.e. $\phi : S_1 \to S_2$ is conformal if $\exists$ function $u : S_1 \to \mathbb{R}$ s.t.

$$g_2(D\phi_p(X), D\phi_p(Y)) = e^{2u(p)} g_1(X, Y)$$

for all $X, Y \in T_pS_1$

Isothermal Coordinates

Fact: Conformality is very flexible.

Theorem: Let S be a surface. For every $p \in S$ there exists an isothermal parametrization for a neighbourhood of p.

- This means that there exists $\mathcal{U} \subseteq \mathbb{R}^2$ and $\mathcal{V} \subseteq S$ containing p, a map $\phi : \mathcal{U} \to \mathcal{V}$ and a function $u : \mathcal{U} \to \mathbb{R}$ so that

$$g := [D\phi_x]^\top D\phi_x = \begin{pmatrix} e^{2u(x)} & 0 \\ 0 & e^{2u(x)} \end{pmatrix} \qquad \forall\, x \in \mathcal{U}$$

- This is proved by solving a fully determined PDE.

Corollary: Every surface is locally conformally planar.

Corollary: Any pair of surfaces is locally conformally equivalent.

A Connection to Complex Analysis

The existence of isothermal parameters can be re-phrased in the language of complex analysis.

- Replace $\mathbb{R}^2$ with $\mathbb{C}$.
- Now every $p \in S$ has a neighbourhood that is holomorphic to a neighboorhood of $\mathbb{C}$.
- The fact that the metric is isothermal is key — multiplication by $\sqrt{-1}$ in $\mathbb{C}$ is equivalent to rotation by $\pi/2$ in S.
- S becomes a complex manifold.

Fact: The connection to complex analysis is very deep!

The Uniformization Theorem

What happens globally?

Uniformization Theorem:

Let S be a 2D compact abstract surface with metric g. Then S possesses a metric $\bar{g}$ conformal to g with constant Gauss curvature $+1, -1$ or 0.

Furthermore, S is conformal to a model space which is (the quotient by a finite group of self-conformal maps of) one of the following:

- The sphere with its standard metric if S has genus zero.
- The plane with its standard metric if S has genus one.
- The unit disk with the Poincaré metric if S has genus > 1.

The Gauss-Bonnet Formula

Useful formula: The Gauss curvature transforms as follows under a conformal map:

$$g_2 = e^{2u} g_1 \quad \Longrightarrow \quad K_2 = e^{2u}\big(-\Delta_1 u + K_1 \big)$$

Consequence: The Gauss-Bonnet formula implies that the sign of the uniformized curvature depends on topology.

$$\begin{aligned} const. \times \text{Area}(S) &= \int_S K_2 \, dA_2 \\ &= \int_S e^{2u}\big(-\Delta_1 u + K_1 \big) \times e^{-2u} \, dA_1 \\ &= \int_S K_1 \, dA_1 \\ &= 2\pi\chi(S) \end{aligned}$$

Genus-One Surfaces

If S has genus zero then it is conformal to the sphere.

Key fact: The map $\phi : S \to \mathbb{S}^2$ is not unique.

- $\mathbb{S}^2$ is conformal to $\mathbb{C} \cup \{\infty\}$ by stereographic projection.
- The set $\mathcal{M}$ of conformal self-maps of $\mathbb{C} \cup \{\infty\}$ is the set of all Möbius transformations of the complex plane.
- Thus any two conformal maps $\phi_1, \phi_2 : S \to \mathbb{S}^2$ satisfy

$$\sigma \circ \phi_1 \circ \phi_2^{-1} \circ \sigma^{-1} = m \in \mathcal{M}$$

Another Key Fact: Conformal maps to the sphere minimize the Dirichlet energy $\mathcal{E}_D(\phi) := \int_S \|D\phi\|_F^2 \, dA$.

- Thus we can find ϕ by flowing down the energy gradient.
- Must impose a condition to ensure convergence to a unique solution — e.g. $\int_S \phi = 0$.

Higher-Genus Surfaces

For genus > 0 surfaces, there is more than one possible candidate for the $\underbrace{\text{“target surface in the Uniformization Theorem and its metric”}}_{\text{The conformal structure of } S}$.

Def: The set $\mathcal{T}_S$ of conformal structure of S is called the Teichmüller space of S and is an abstract manifold of dimension

$$dim(\mathcal{T}_S) = \begin{cases} 2 & genus = 1 \\ 6g - 6 & genus = g > 1 \end{cases}$$

A parametrization of $\mathcal{T}_S$ is provided by holomorphic differentials:

- These are related to harmonic one-forms on S.
- The natural coordinates of $\mathcal{T}_S$ are the values of the line integrals of these differentials around homology generators of S.