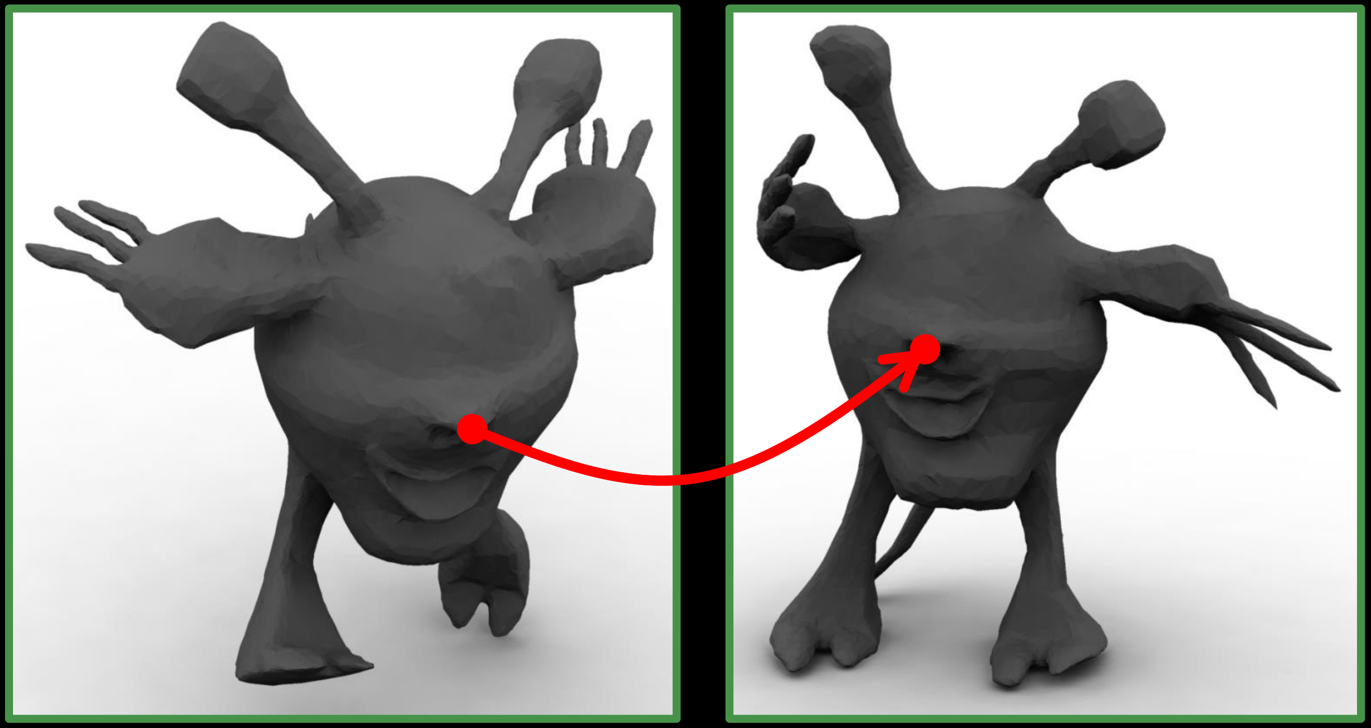
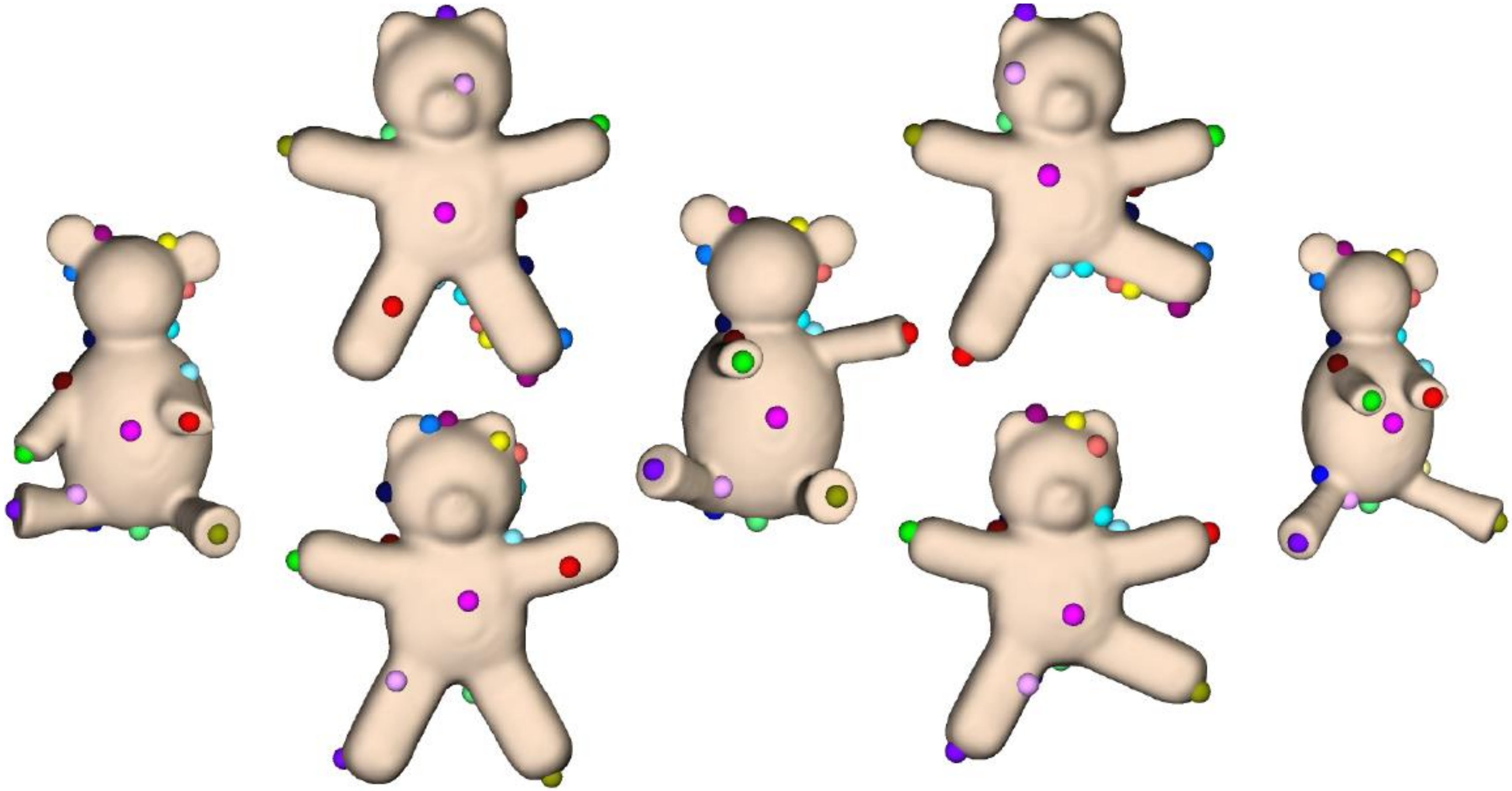


Isometry Invariance and Spectral Techniques

CS 468, Spring 2013
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher

Instances of "Same" Shape



http://graphics.stanford.edu/projects/lgl/papers/nbwyg-oaicsm-11/nbwyg-oaicsm-11.pdf

Need to understand deformations

Deformation-Invariant Applications

- Segmentation
- Symmetry detection
- Global shape description
- Retrieval
- Recognition
- Feature extraction
- Alignment
- …

Isometry

[ahy-**som**-i-tree]**:**

Bending without stretching.

Lots of Interpretations

Global isometry

$$d_1(x,y) = d_2(f(x), f(y))$$

Local isometry

$$g_1 = f^* g_2$$

$$g_1(v,w) = g_2(f_* v, f_* w)$$

Intrinsic Techniques

http://www.revedreams.com/crochet/yarncrochet/nonorientable-crochet/

Isometry invariant

Isometry Invariance: Hope

http://www.flickr.com/photos/melvinvoskuijl/galleries/72157624236168459

Isometry Invariance: Reality

http://www.4tnz.com/content/got-toilet-paper

Few shapes *can* deform isometrically

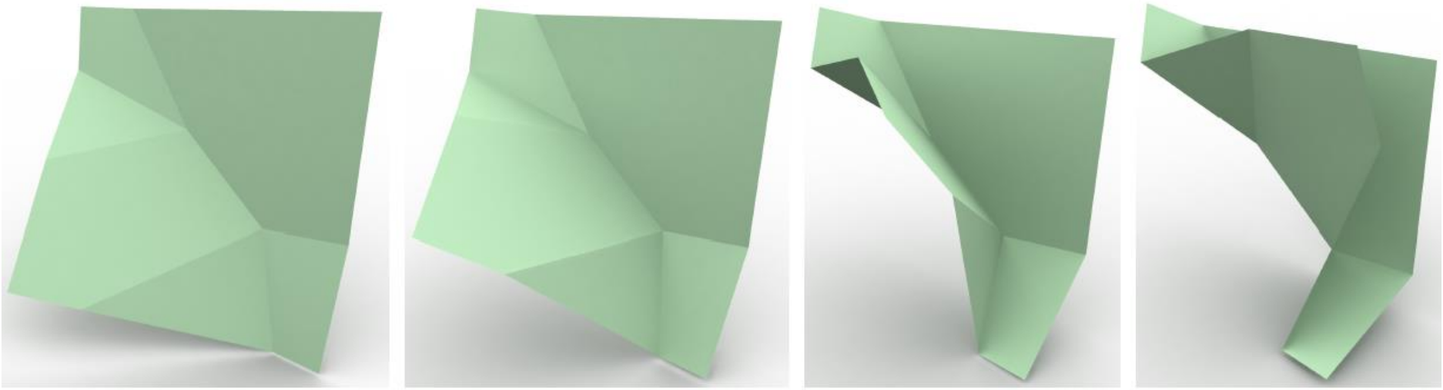
Isometry Invariance: Reality

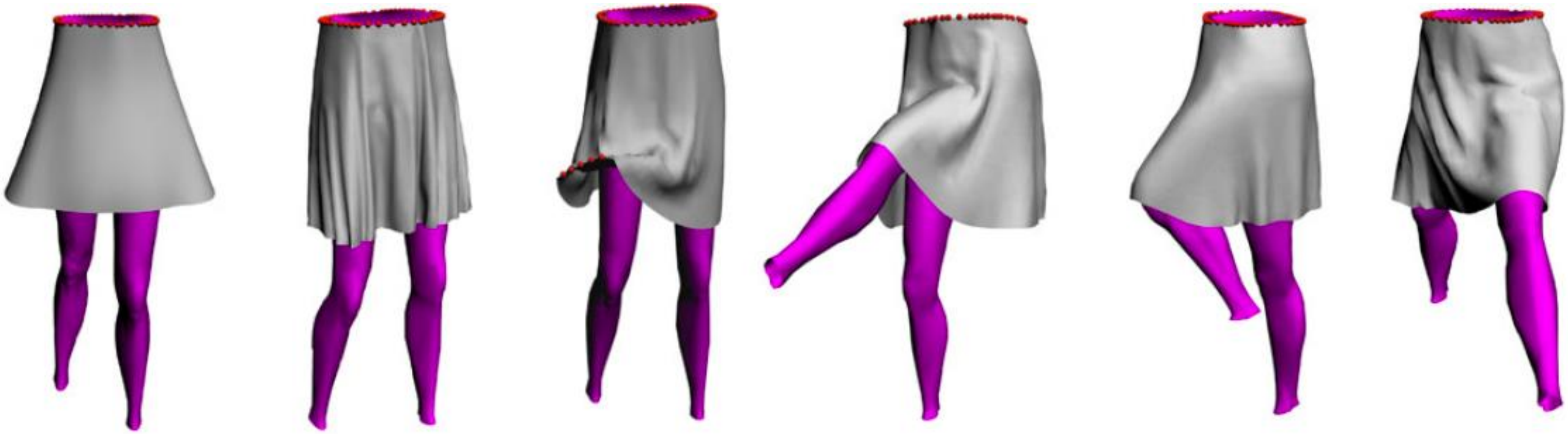
Behavior for approximate isometry is important, too!

http://www.4tnz.com/content/got-toilet-paper

Few shapes *can* deform isometrically

Separate Thread in DDG





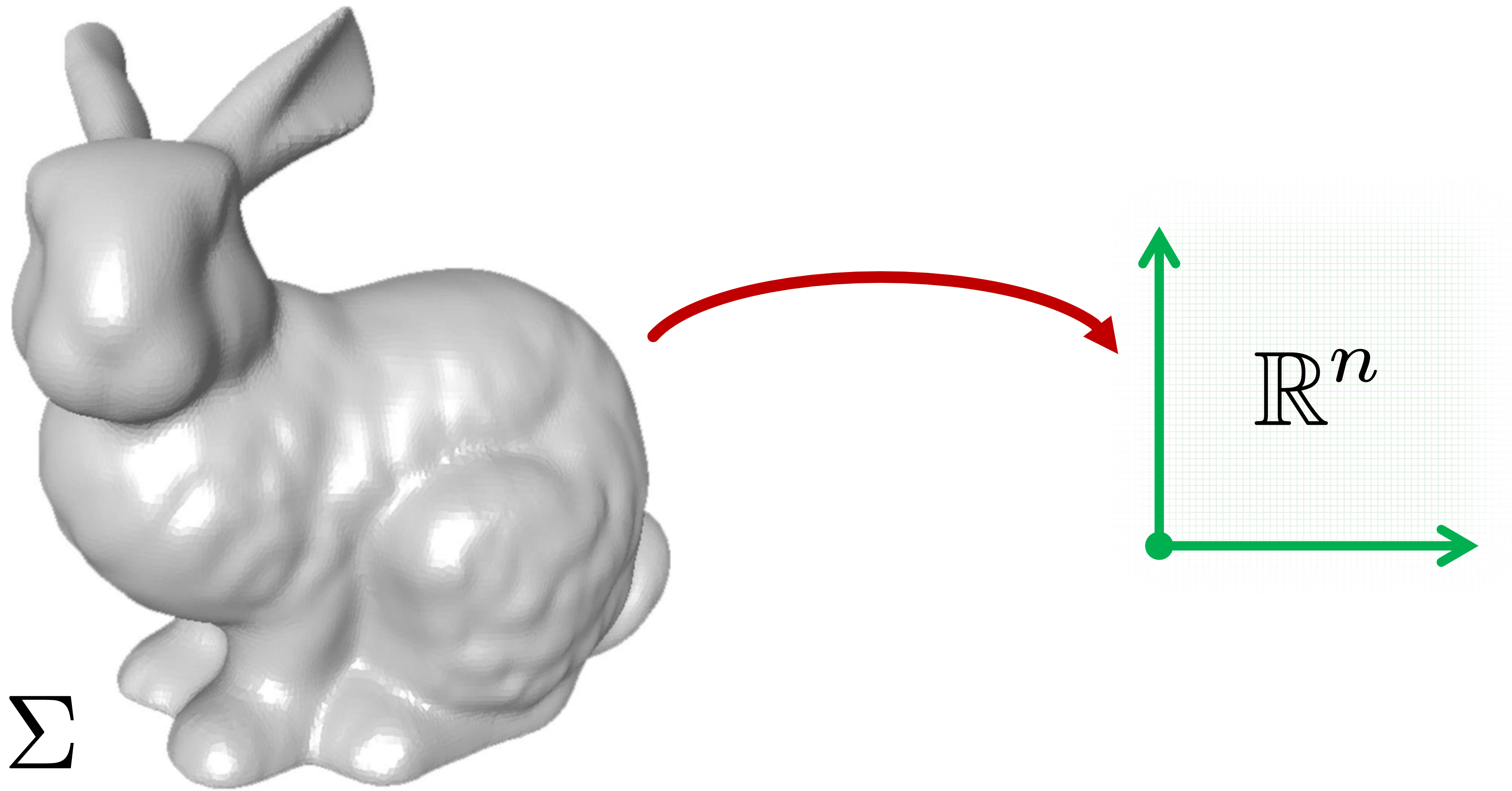
http://www.stanford.edu/~justso1/assets/discrete_developables.pdf http://link.springer.com/article/10.1007%2Fs00371-010-0467-5

Developable surfaces and origami

Rigidity

Most surfaces cannot deform isometrically whatsoever.

Example Task: Shape Descriptors



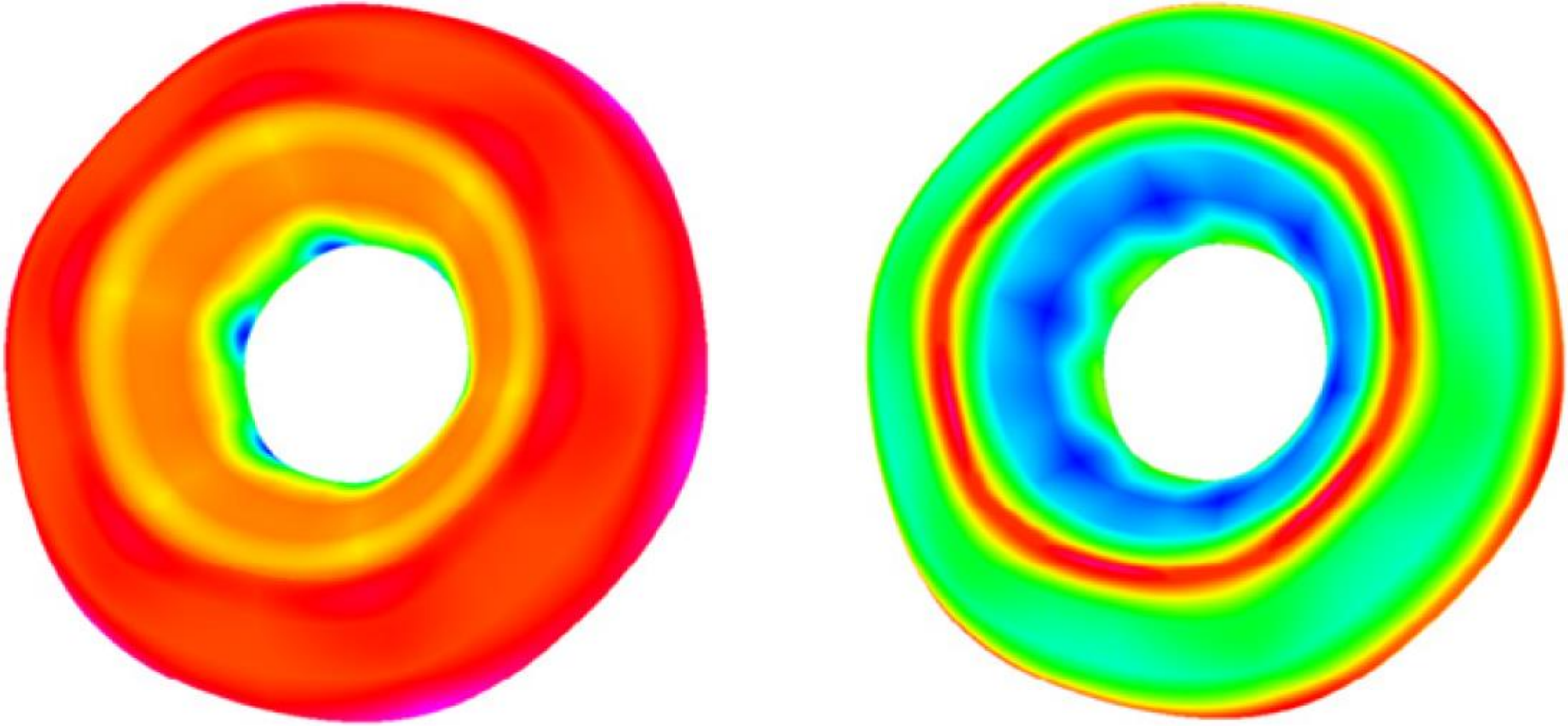
http://liris.cnrs.fr/meshbenchmark/images/fig_attacks.jpg

Pointwise quantity

Descriptor Tasks

- Characterize **local geometry**
- Describe a point's "**role**" on a surface

Descriptors We've Seen Before



$K = \kappa_1 \kappa_2$ $\quad$ $H = 1/2(\kappa_1 + \kappa_2)$

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian and mean curvature

Desirable Properties

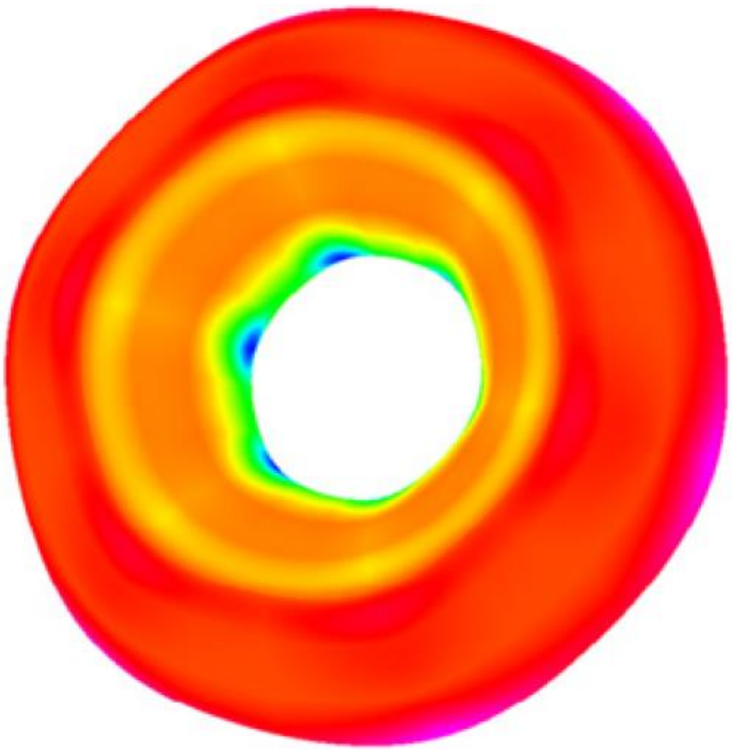
- **Distinguishing**
  Provides useful information about a point
- **Stable**
  Numerically and geometrically
- **Intrinsic**
  No dependence on embedding

Intrinsic Descriptors

Invariant under

- Rigid motion
- Bending without stretching

Intrinsic Descriptor



$$K = \kappa_1 \kappa_2$$

Theorema Egregium

("Totally Awesome Theorem"):

Gaussian curvature is intrinsic.

http://www.sciencedirect.com/science/article/pii/S0010448510001983

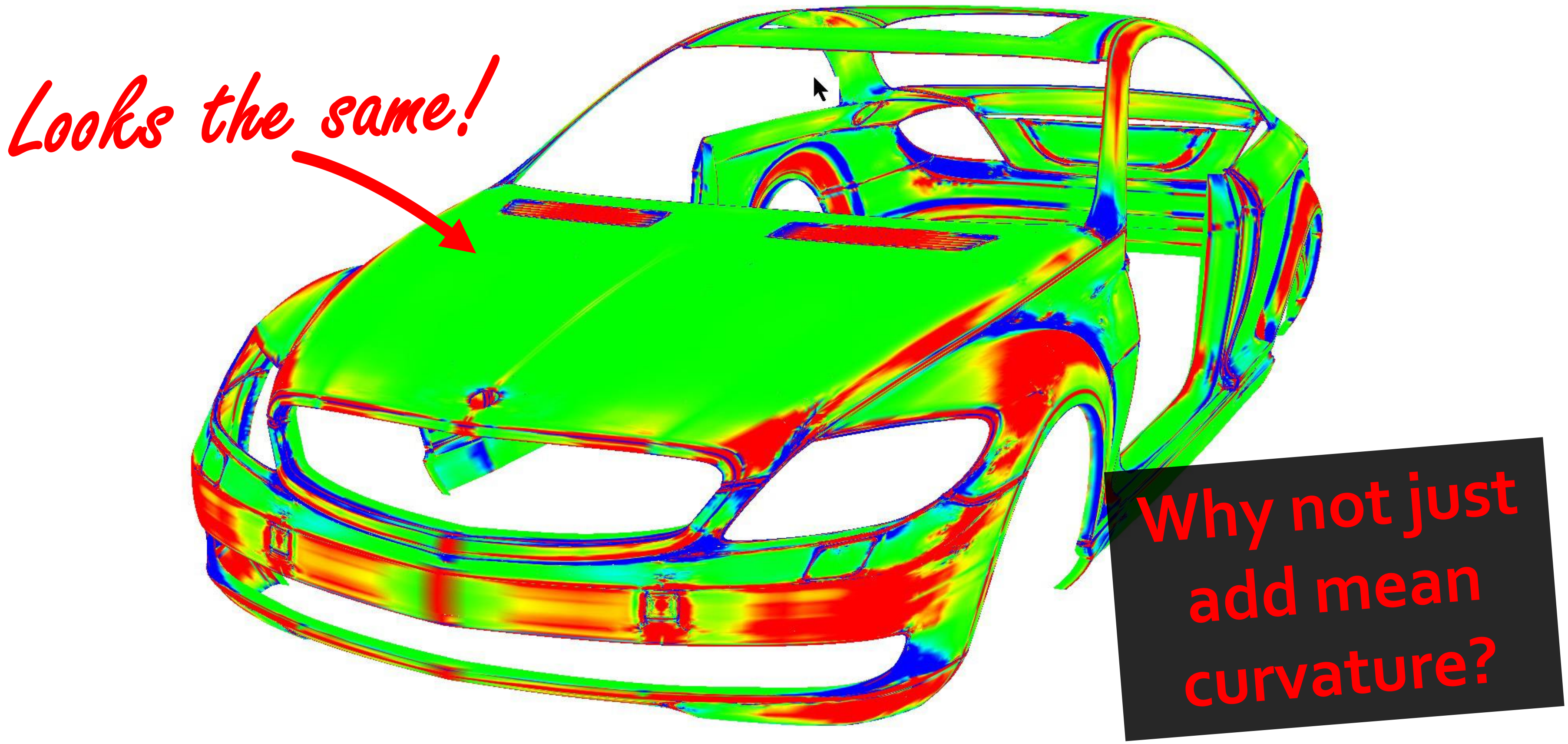
Gaussian curvature

End of the Story?

$$K = \kappa_1 \kappa_2$$

Second derivative quantity

End of the Story?



http://www.integrityware.com/images/MerceedesGaussianCurvature.jpg

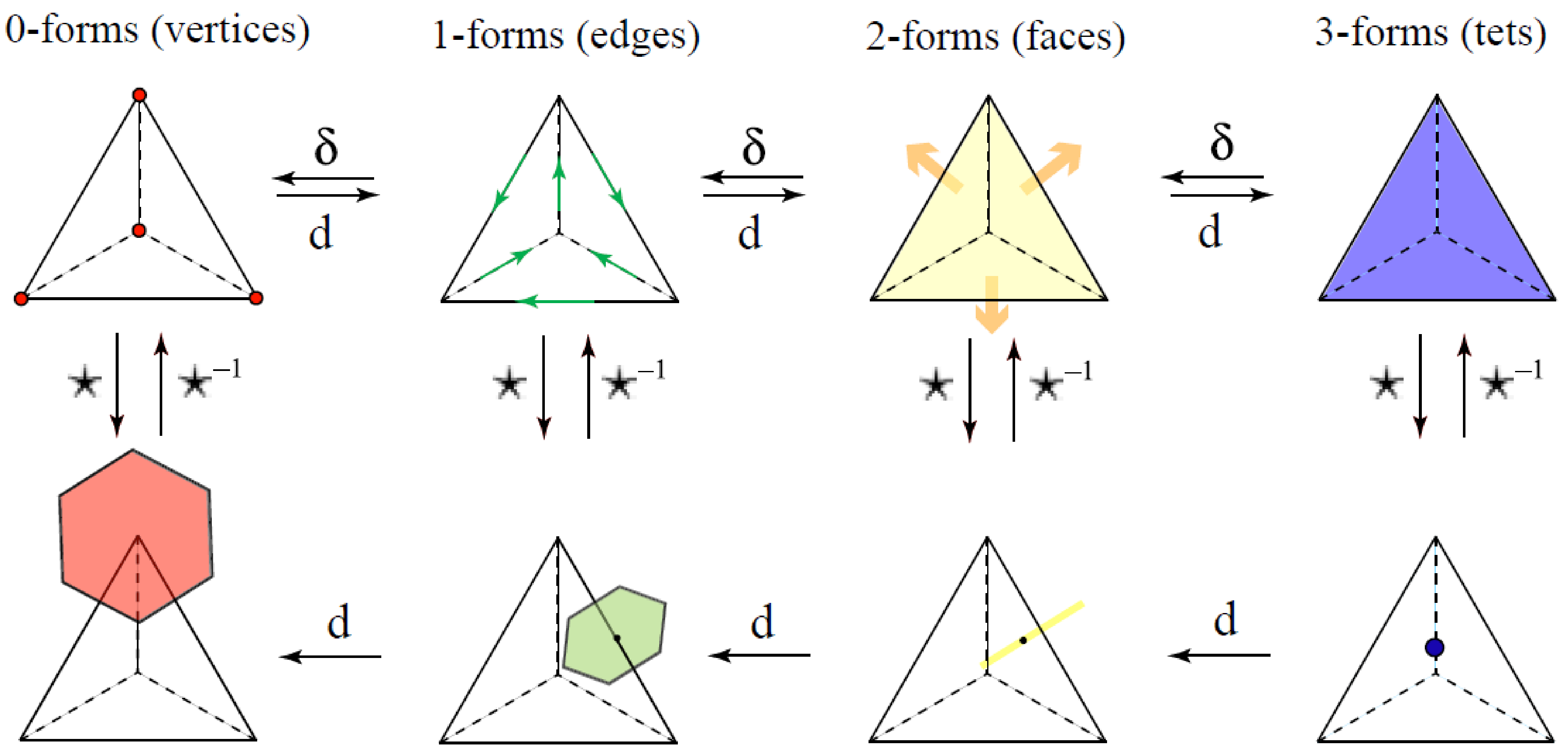
Nonunique

Desirable Properties

Incorporates **neighborhood information** in an intrinsic fashion

Stable under small deformation

Recall: (Discrete) deRham Complex



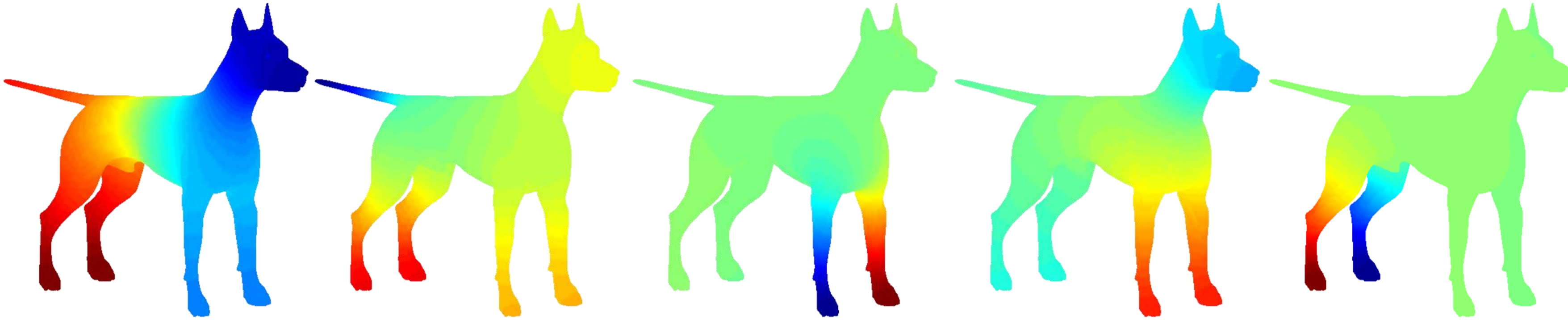
http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf

Intrinsic!

Recall:

Hodge Laplacian

$$\Delta = d \star d \star + \star d \star d$$



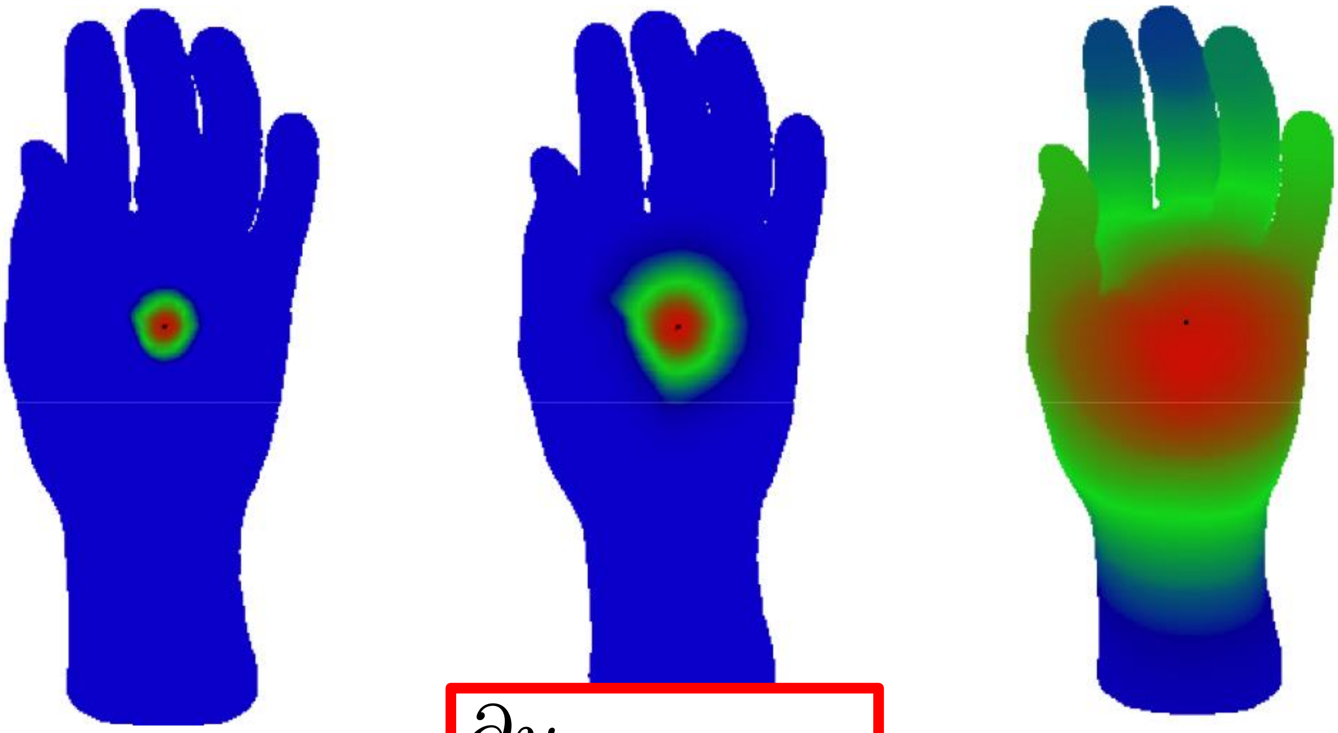
$\Delta\phi_1 = \lambda_1\phi_1$ $\Delta\phi_2 = \lambda_2\phi_2$ $\Delta\phi_3 = \lambda_3\phi_3$ $\Delta\phi_4 = \lambda_4\phi_4$ $\Delta\phi_5 = \lambda_5\phi_5$ $\ldots$

$(\Delta\phi_0 = 0)$

Intrinsic operator

Recall:

Connection to Physics



$$\frac{\partial u}{\partial t} = -\Delta u$$

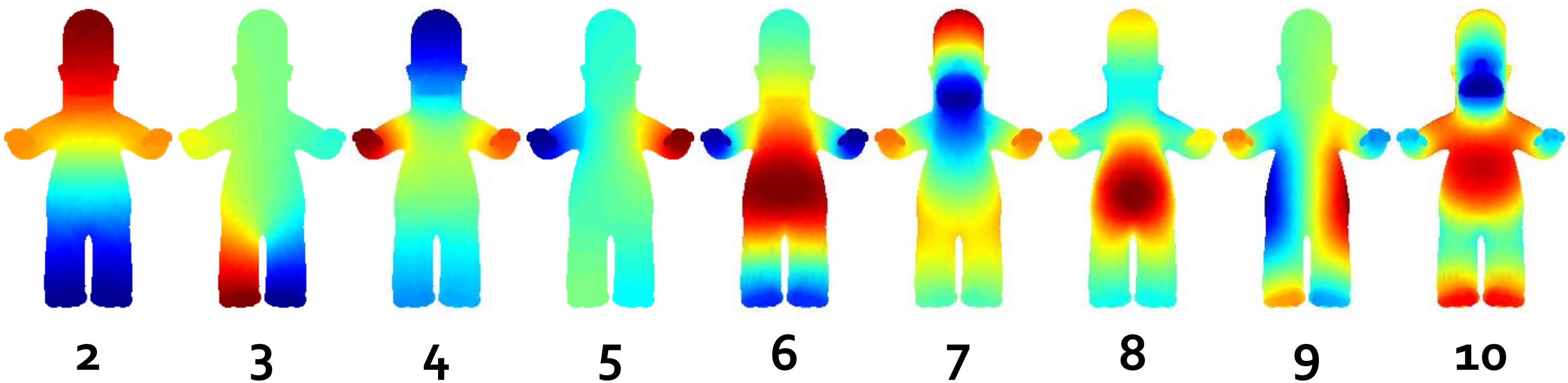
http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Intrinsic Statement

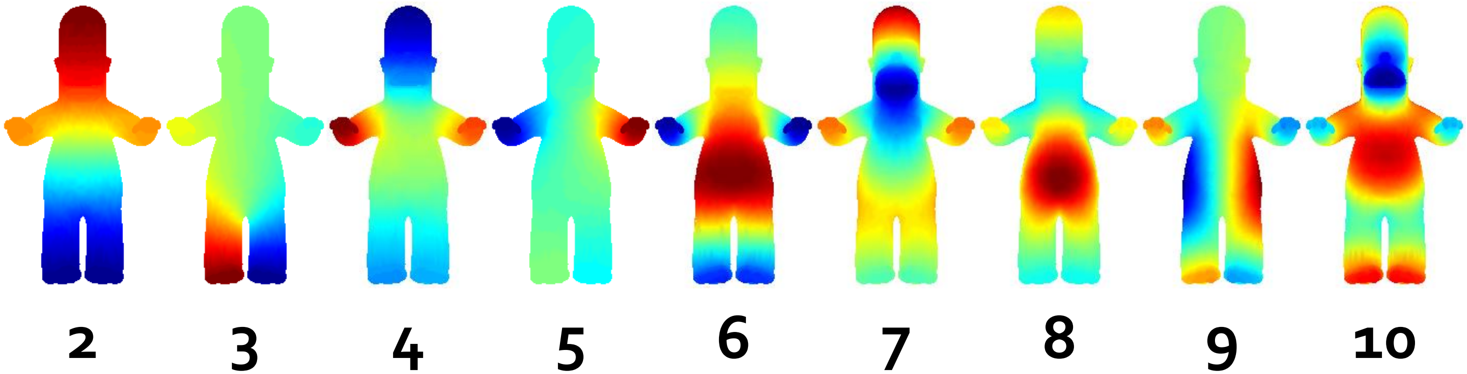
Heat diffusion patterns are not affected if you bend a surface.

Global Point Signature



$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \cdots \right)$$

Global Point Signature

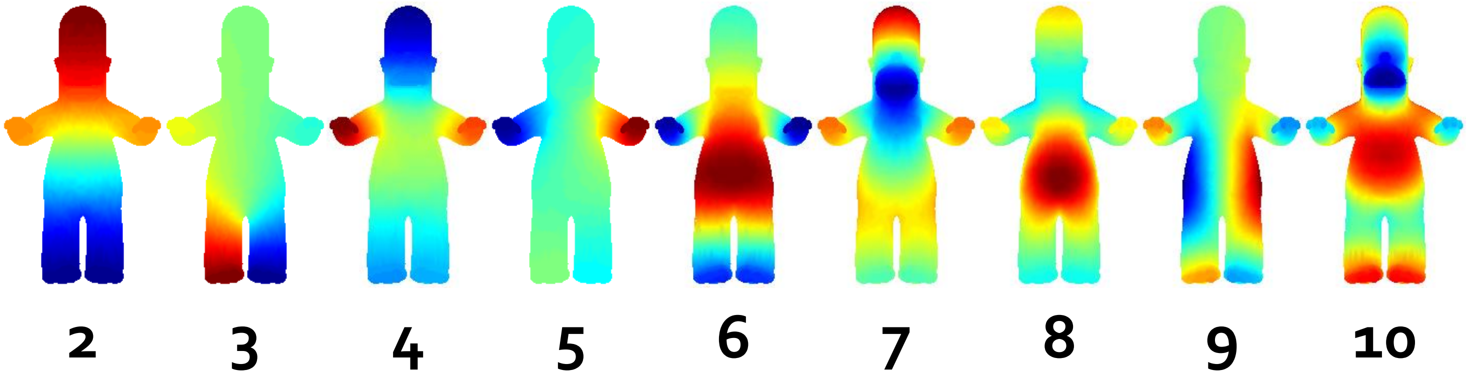


$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \cdots \right)$$

If surface does not self-intersect, neither does the GPS embedding.

Proof: Laplacian eigenfunctions span $L^2(\Sigma)$; if GPS(*p*)=GPS(*q*), then all functions on Σ would be equal at *p* and *q*.

Global Point Signature



$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

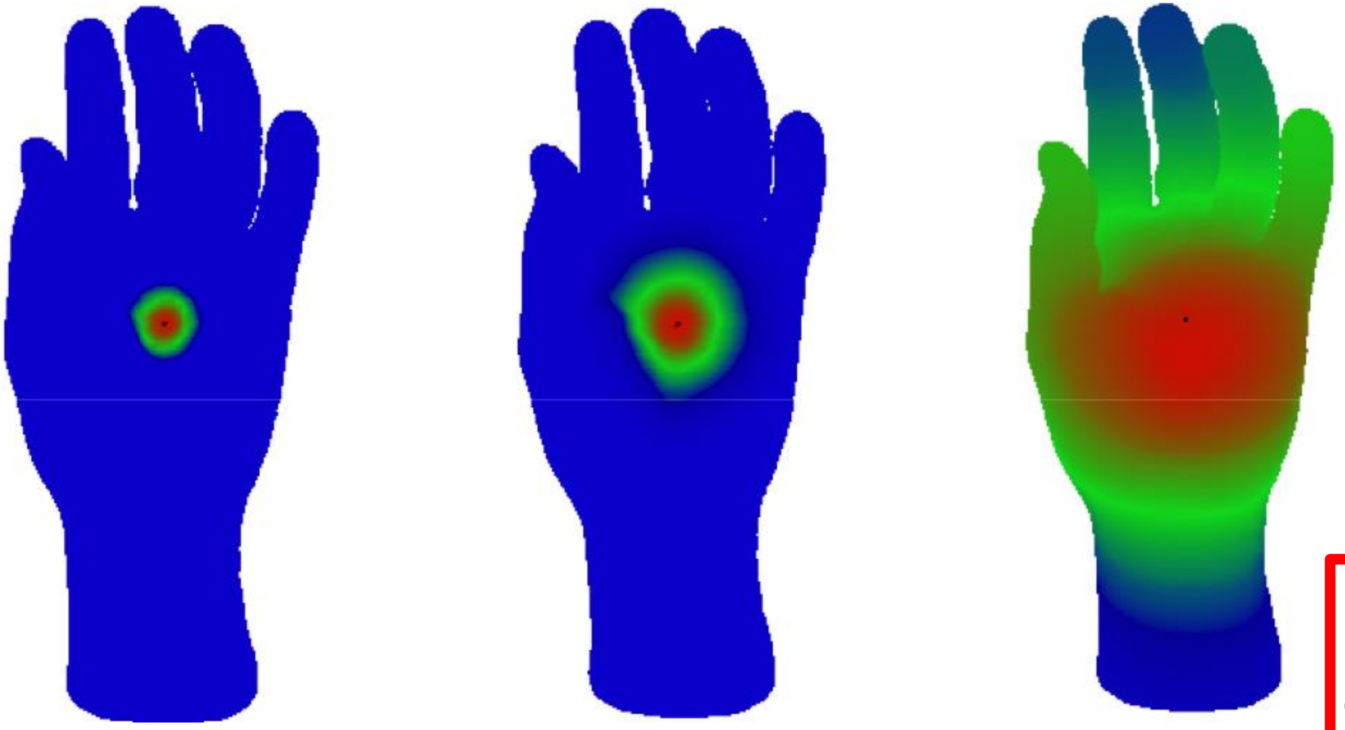
GPS is isometry-invariant.

Proof: Comes from the Laplacian.

Drawbacks of GPS

- Assumes unique λ's
- Potential for eigenfunction "switching"
- Nonlocal feature

PDE Applications of the Laplacian

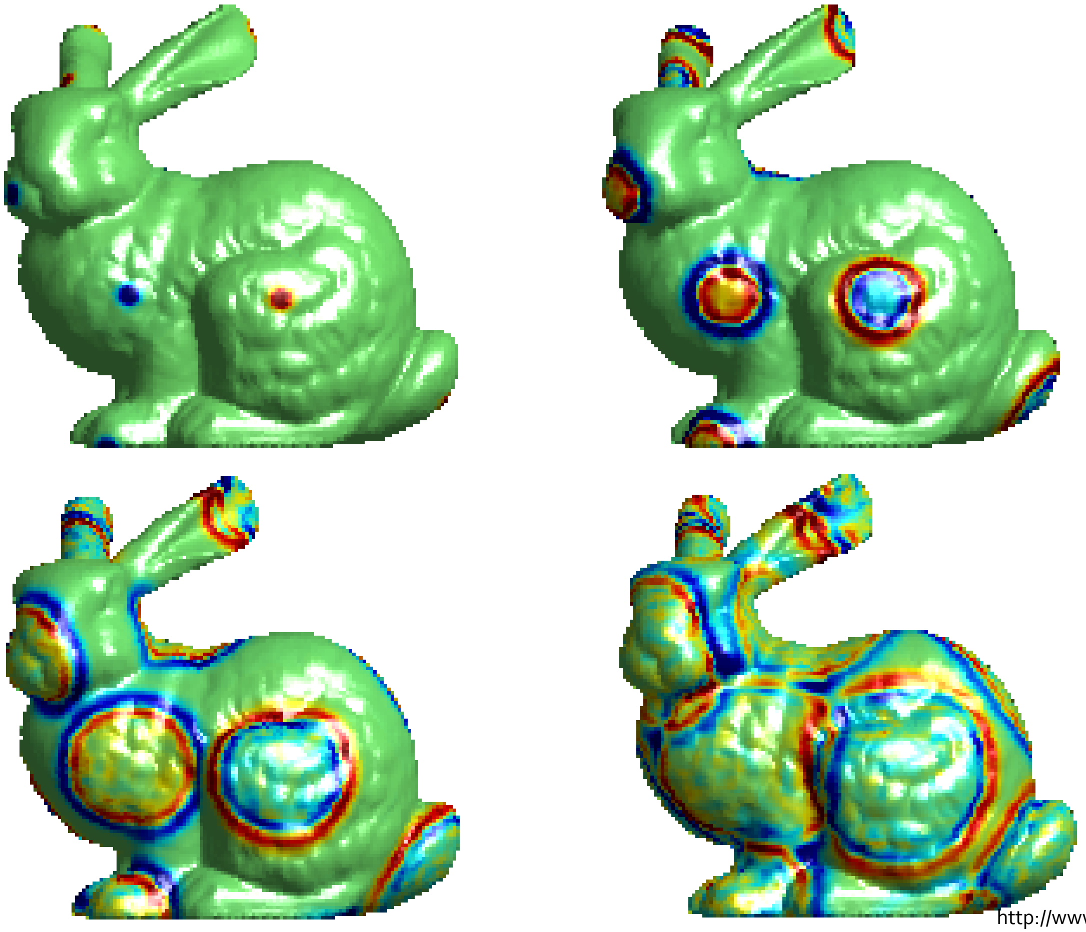


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

PDE Applications of the Laplacian



$$\frac{\partial^2 u}{\partial t^2} = -i\Delta u$$

http://www.ceremade.dauphine.fr/~peyre/numerical-tour/tours/meshproc_5_pde/index_06.png

Wave equation

PDE Applications of the Laplacian

Use this behavior to characterize shape.

$$\frac{\partial^2 u}{\partial t^2} = -i\Delta u$$

http://www.ceremade.dauphine.fr/~peyre/numerical-tour/tours/meshproc_5_pde/index_06.png

Wave equation

Solutions in the LB Basis

$$\frac{\partial u}{\partial t} = -\Delta u$$

Heat equation

$$u = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \phi_n(x)$$

$$\left(a_n = \int_{\Sigma} u_0 \cdot \phi_n \; dA \right)$$

Heat Kernel Signature (HKS)

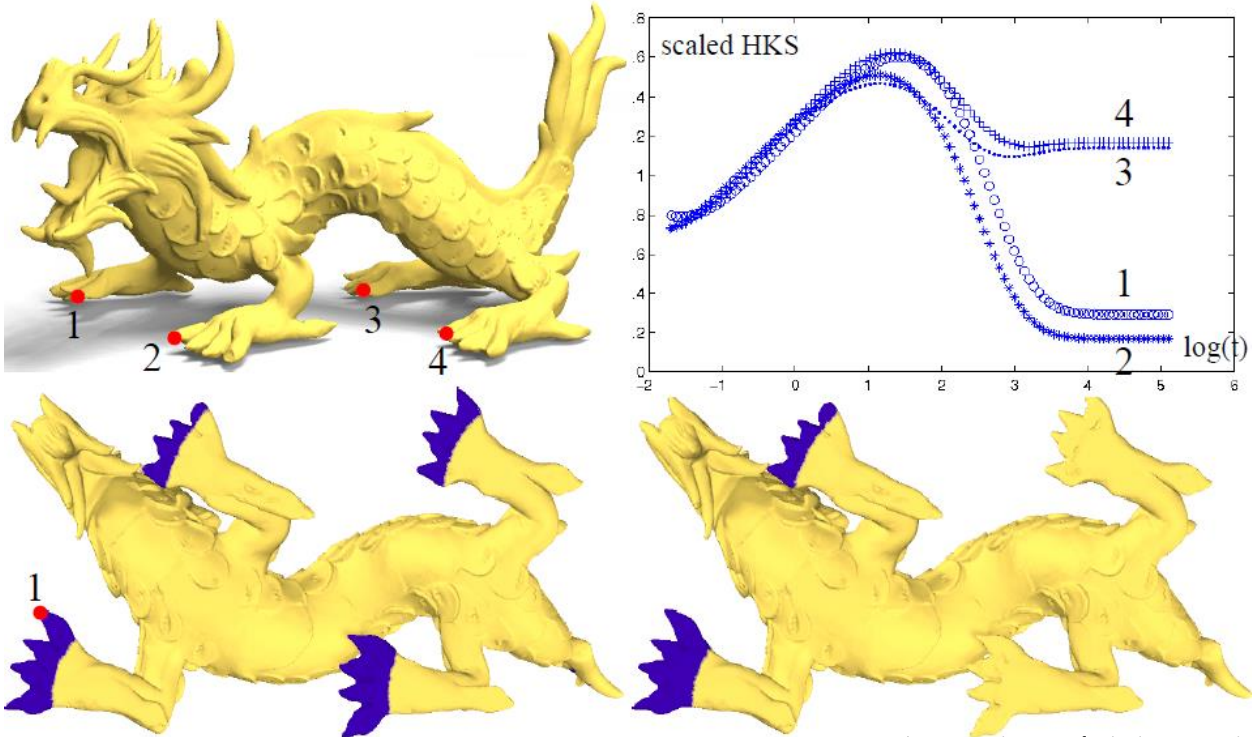
$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Continuous function on [0,∞)

How much heat diffuses from *x* to itself in time *t*?

Heat Kernel Signature (HKS)

$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$



http://graphics.stanford.edu/projects/lgl/papers/sog-hks-09/sog-hks-09.pdf

Heat Kernel Signature (HKS)

$$k_t(x, x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Good properties:

- Isometry-invariant
- Multiscale
- Not subject to switching
- Easy to compute
- Related to curvature at small scales

Heat Kernel Signature (HKS)

$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Bad properties:

- Issues remain with repeated eigenvalues
- Theoretical guarantees require (near-)isometry

Wave Kernel Signature (WKS)

$$WKS(E,x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |\psi_E(x,t)|^2 \, dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_k)^2$$

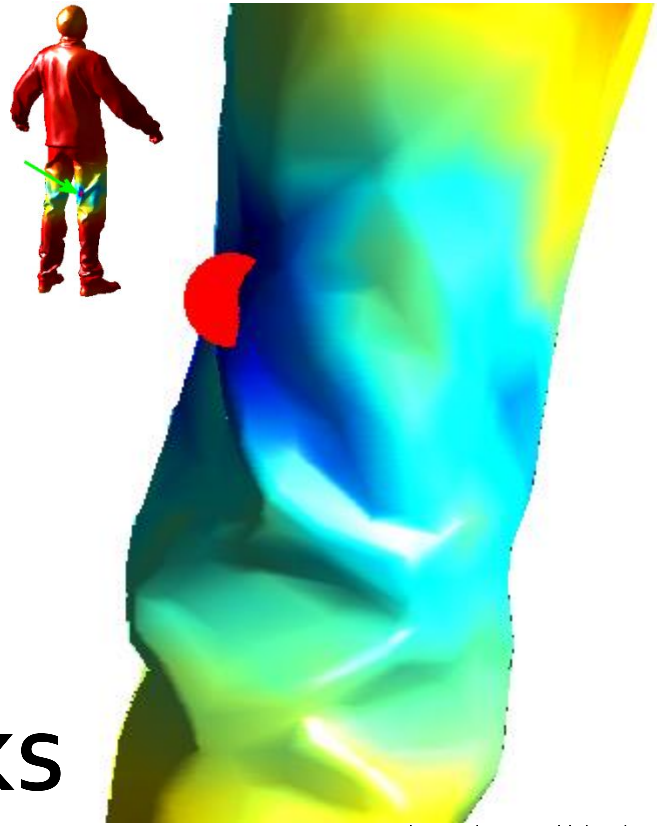
Initial energy distribution

Average probability over time that particle is at *x*.

Wave Kernel Signature (WKS)

$$WKS(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2 \, dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_k)^2$$

HKS



WKS

vision.in.tum.de/_media/spezial/bib/aubry-et-al-4dmod11.pdf

Wave Kernel Signature (WKS)

$$WKS(E,x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |\psi_E(x,t)|^2 \, dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_k)^2$$

Good properties:

- [Similar to HKS]
- Localized in frequency
- Stable under some non-isometric deformation
- Some multi-scale properties

Wave Kernel Signature (WKS)

$$WKS(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2\,dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_k)^2$$

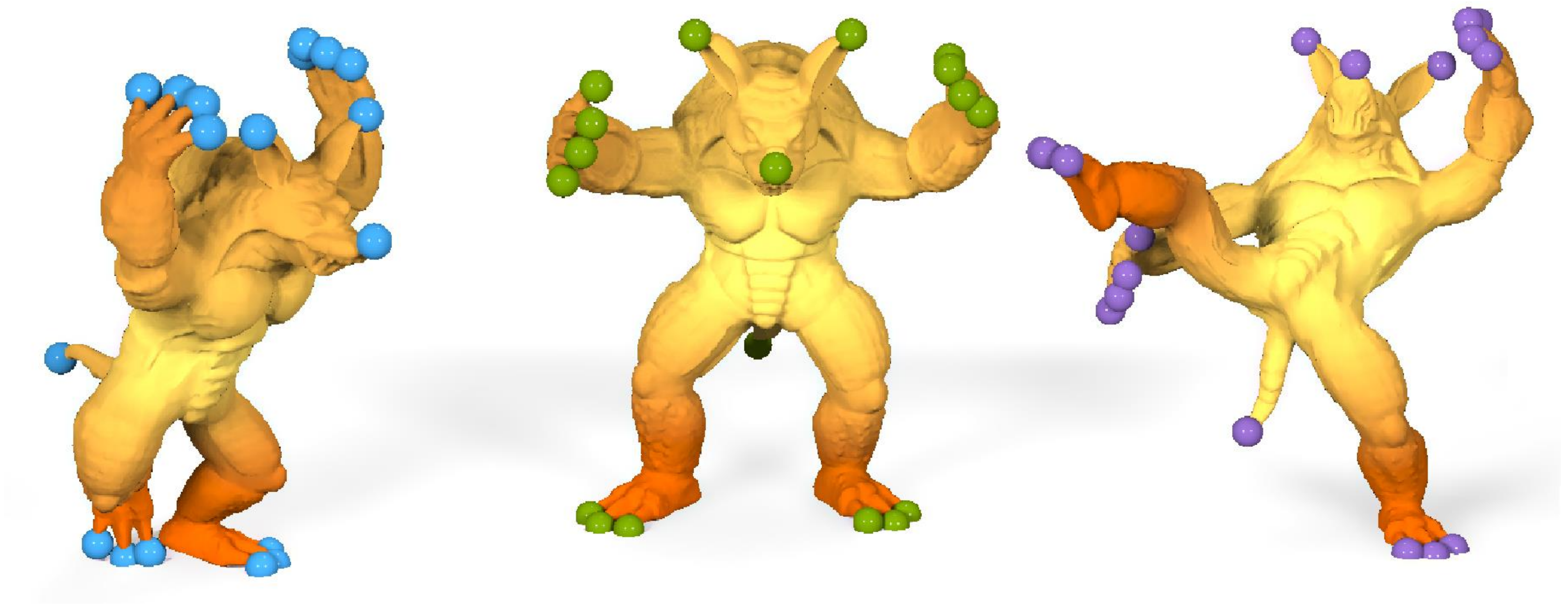
Bad properties:

- [Similar to HKS]
- Can filter out *large*-scale features

Many Others

Lots of *spectral descriptors* in terms of Laplacian eigenstructure.

Application: Feature Extraction

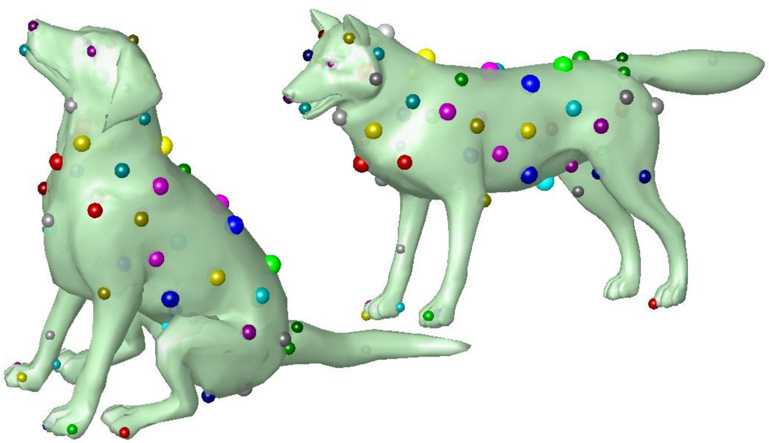


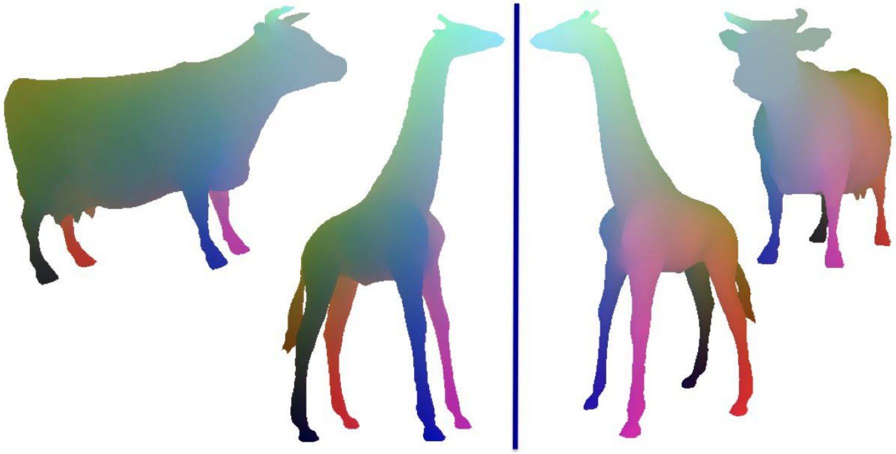
Maxima of $k_t(x,x)$ for large t.

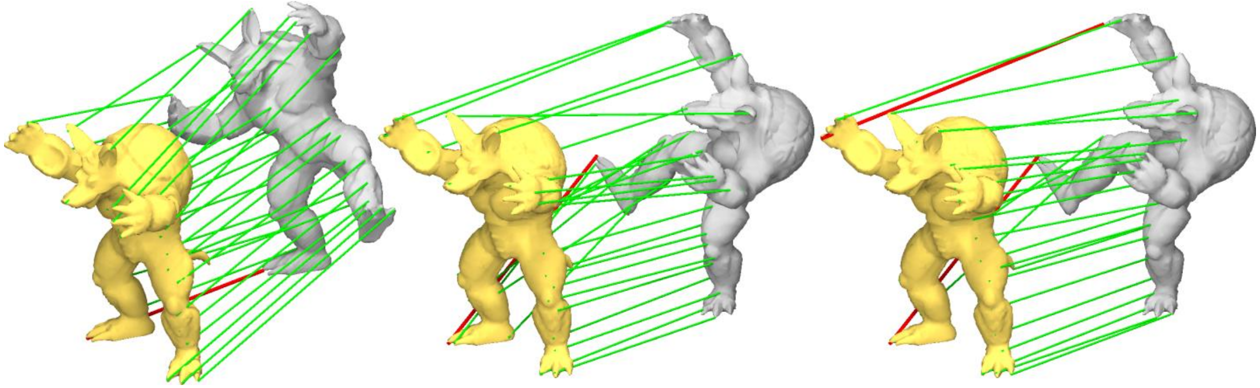
A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion
Sun, Ovsjanikov, and Guibas 2009

Feature points

Related Problem: Correspondence







http://graphics.stanford.edu/projects/lgl/papers/ommg-opimhk-10/ommg-opimhk-10.pdf
http://www.cs.princeton.edu/~funk/sig11.pdf
http://gfx.cs.princeton.edu/pubs/Lipman_2009_MVF/mobius.pdf

Rigid method:

Iterative Closest Point (ICP)

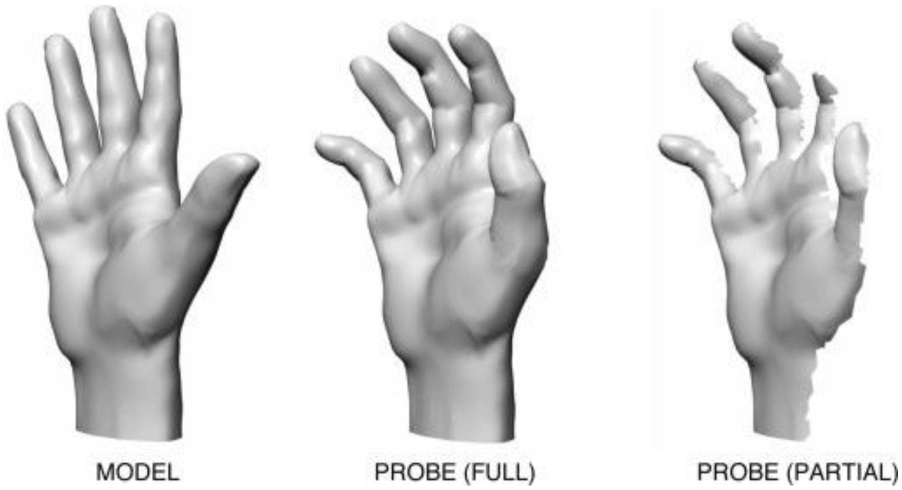
Repeat:

1. For each x_i in X, find closest y_i in Y.
2. Find rigid deformation (R,T) minimizing

$$\sum_i \|(Rx_i + T) - y_i\|$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf
http://www.gris.uni-tuebingen.de/people/staff/bokeloh/gallery/bunny_res1.png

Geodesic-based method: GMDS



Embed samples of one surface directly over another by minimizing a "generalized stress" involving geodesics.

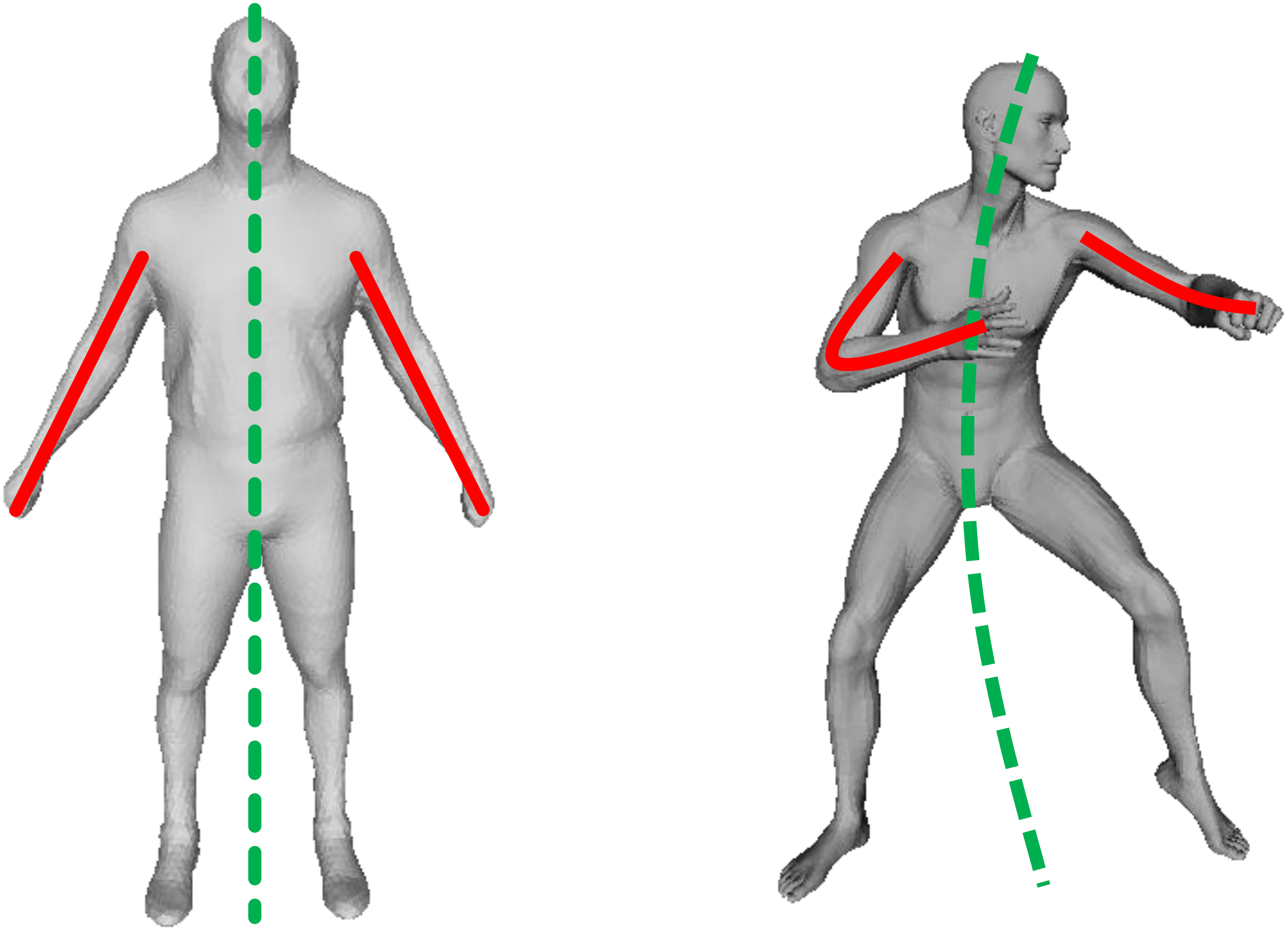
Generalized Multidimensional Scaling
Bronstein, Bronstein, Kimmel 2006

Generalized Multi-Dimensional Scaling

Descriptor Matching

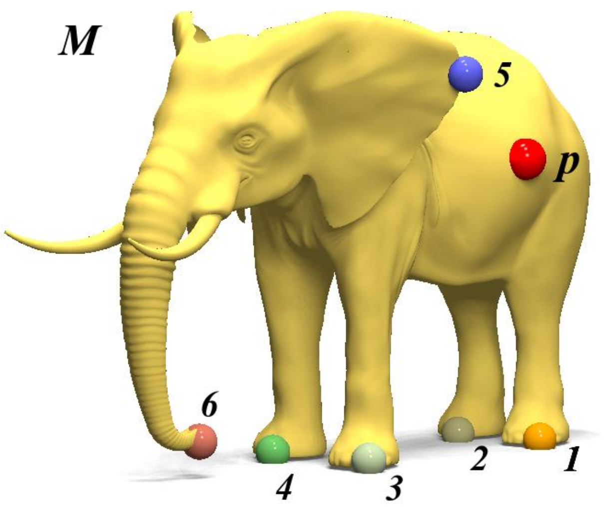
Simply match closest points in descriptor space.

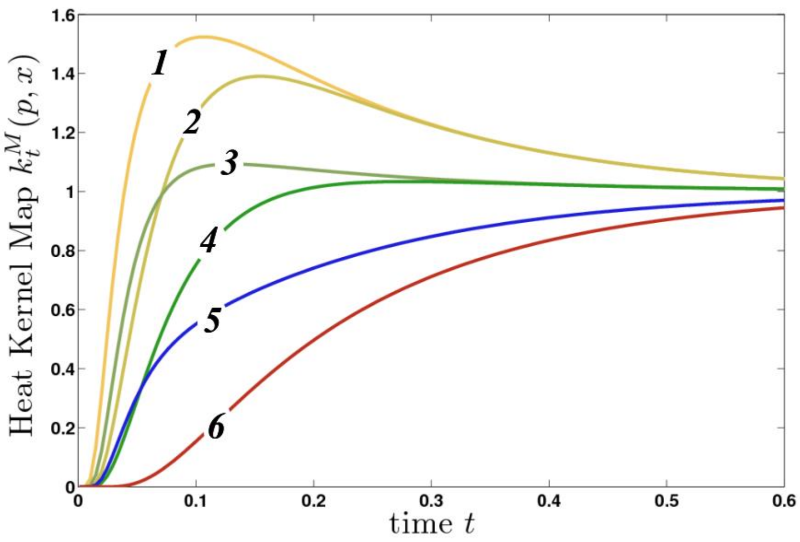
Descriptor Matching Problem



Symmetry

Heat Kernel Map





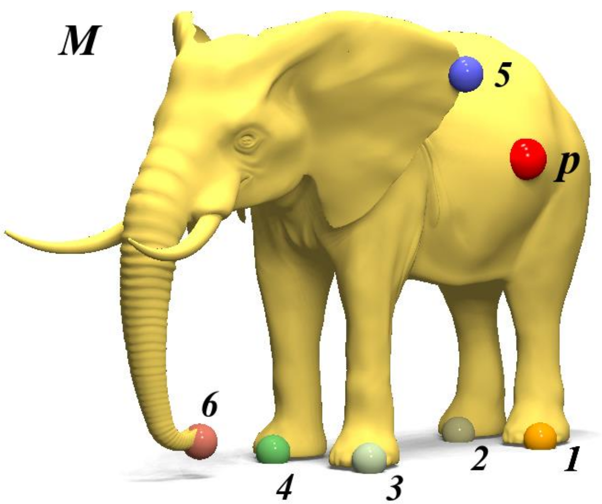
$$HKM_p(x,t) = k_t(p,x)$$

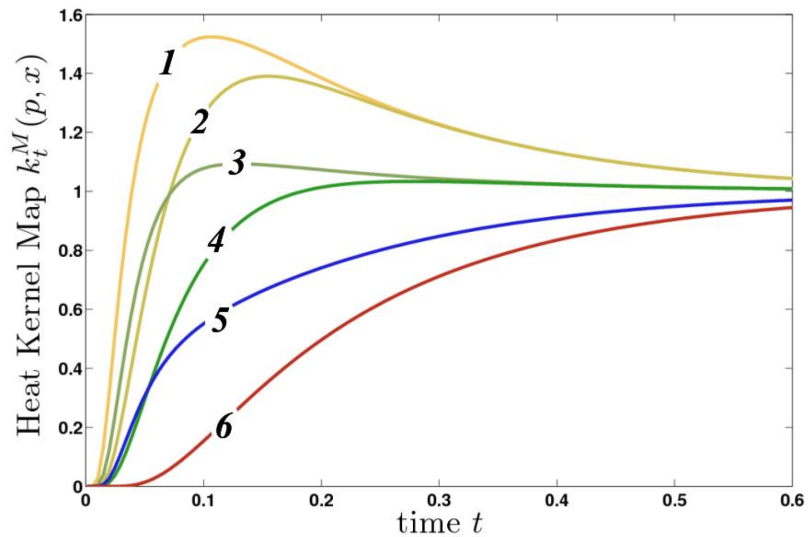
How much heat diffuses from *p* to *x* in time *t*?

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

Heat Kernel Map





$$HKM_p(x,t) = k_t(p,x)$$

Theorem: Only have to match one point!

KNN

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

Recall:

Alternative to Eikonal Equation

Algorithm 1 The Heat Method

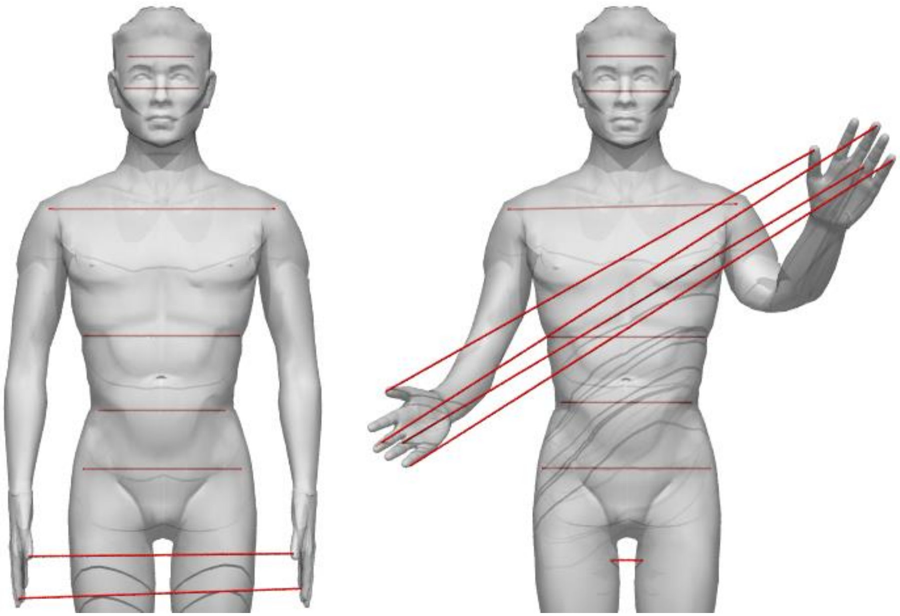
I. Integrate the heat flow $\dot{u} = \Delta u$ for time t.

II. Evaluate the vector field $X = -\nabla u / |\nabla u|$.

III. Solve the Poisson equation $\Delta \phi = \nabla \cdot X$.

Crane, Weischedel, and Wardetzky. "Geodesics in Heat." TOG, to appear.

Self-Map: Symmetry

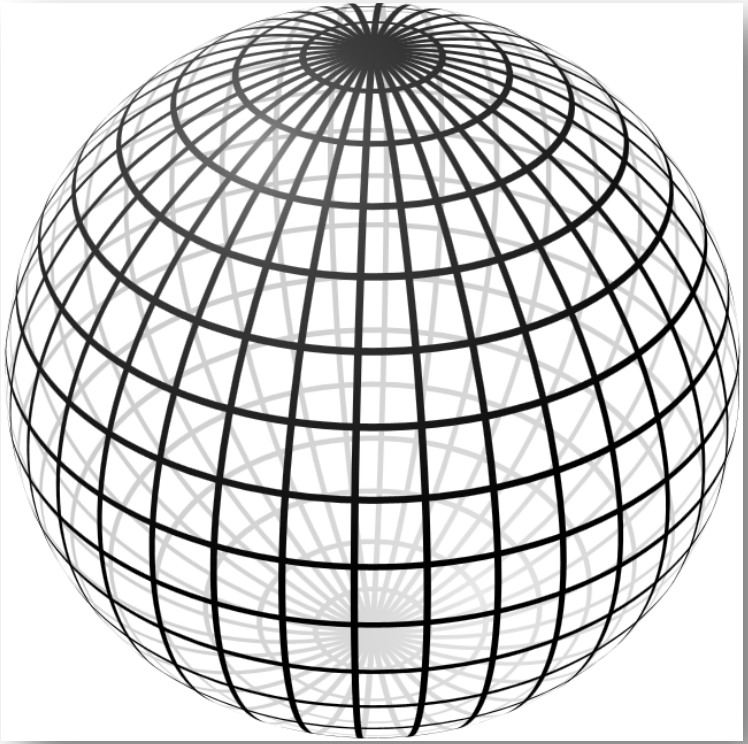


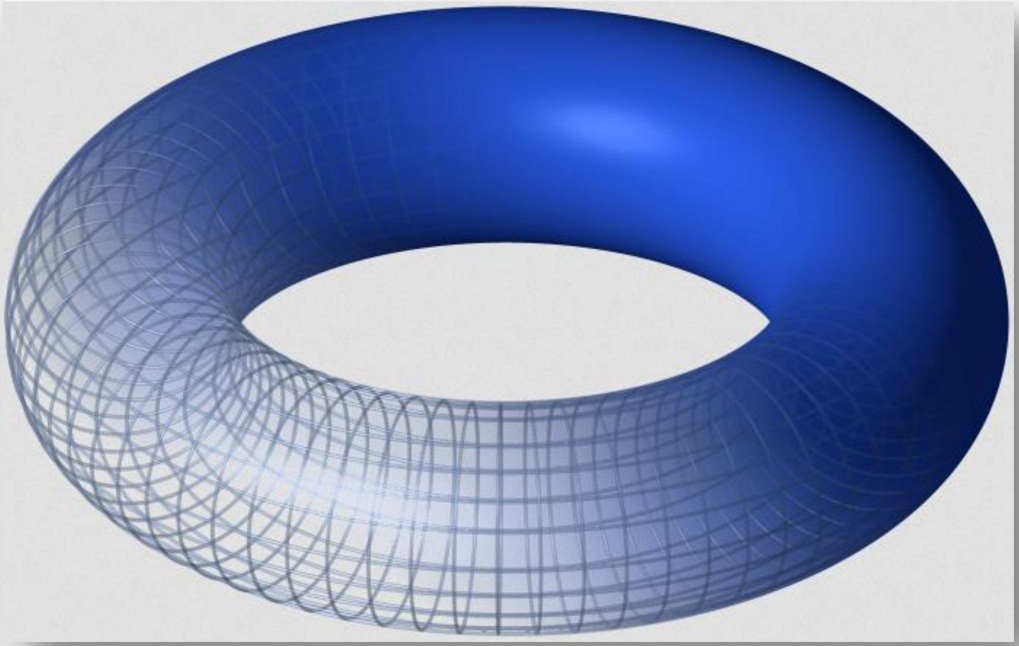
Intrinsic symmetries become extrinsic in GPS space!

Global Intrinsic Symmetries of Shapes
Ovsjanikov, Sun, and Guibas 2008

"Discrete intrinsic" symmetries

Continuous Intrinsic Symmetries?





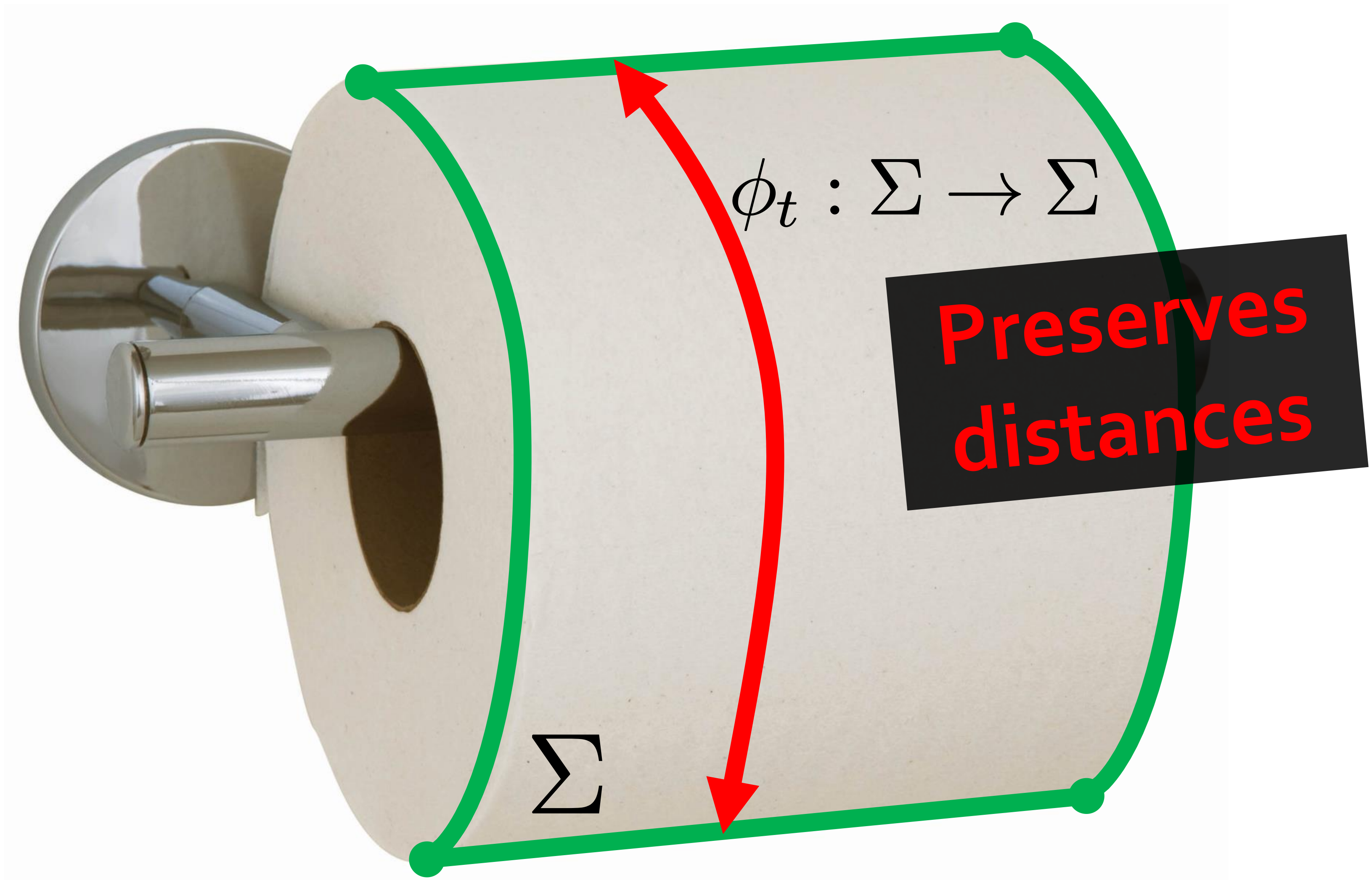
Wikipedia

Intrinsic but not descriptor-based

Continuous Symmetries

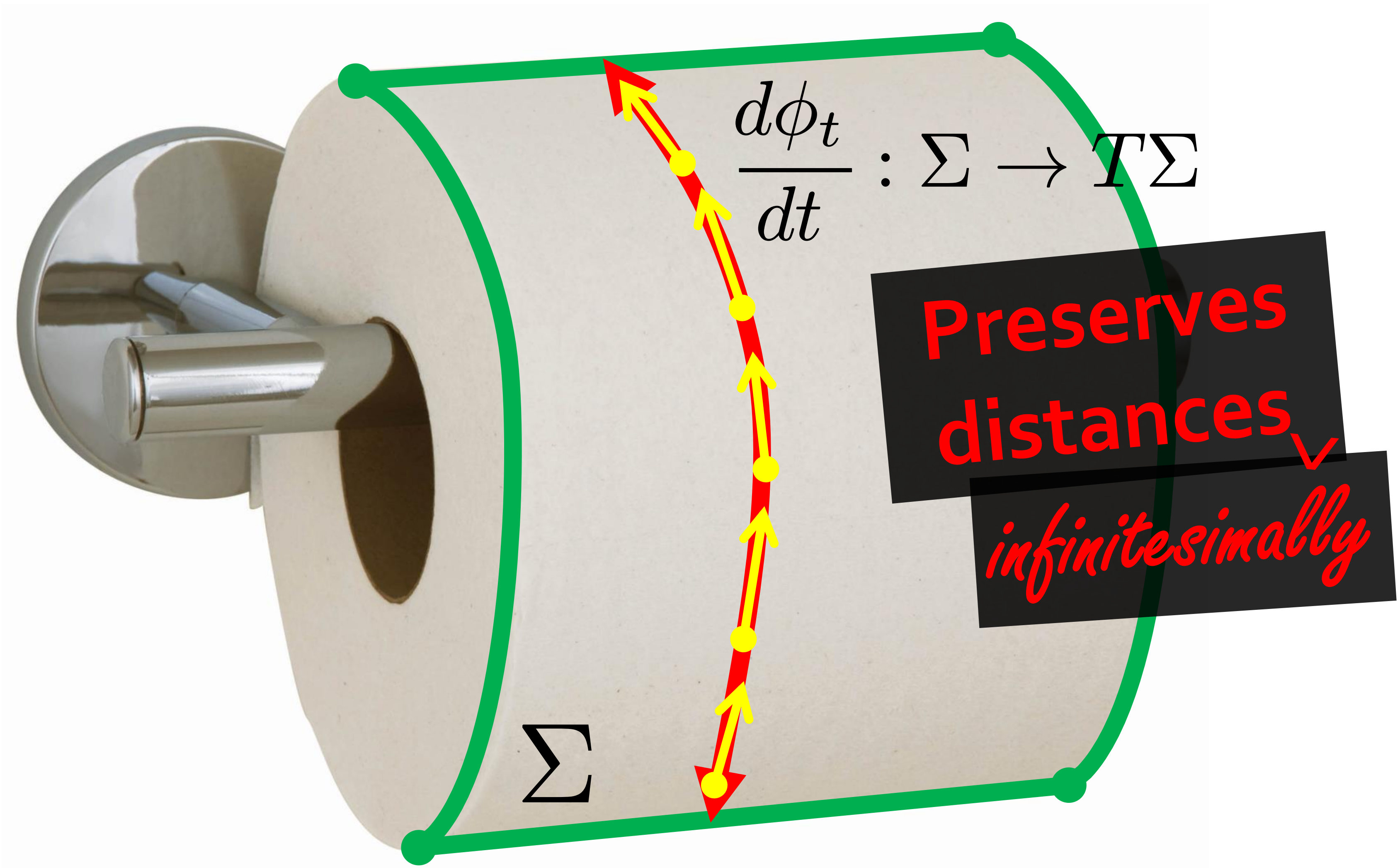
http://www.bradleycorp.com/image/985/9184b_highres.jpg

Continuous Symmetries



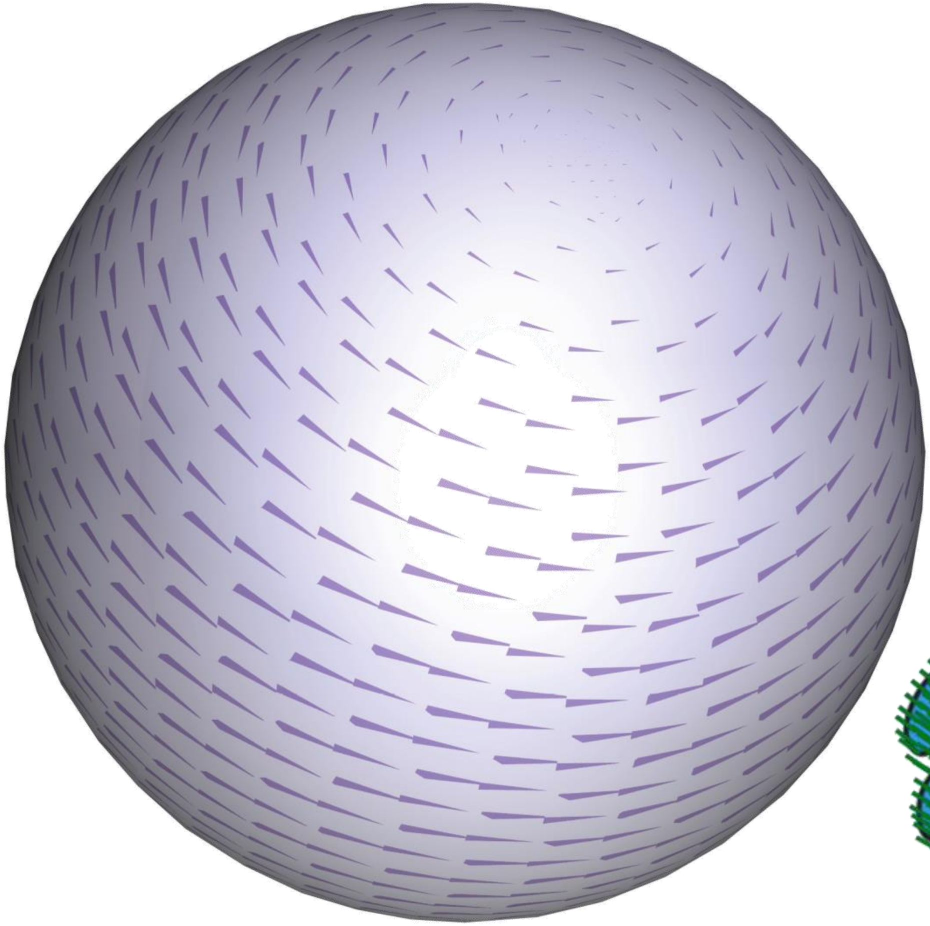
http://www.bradleycorp.com/image/985/9184b_highres.jpg

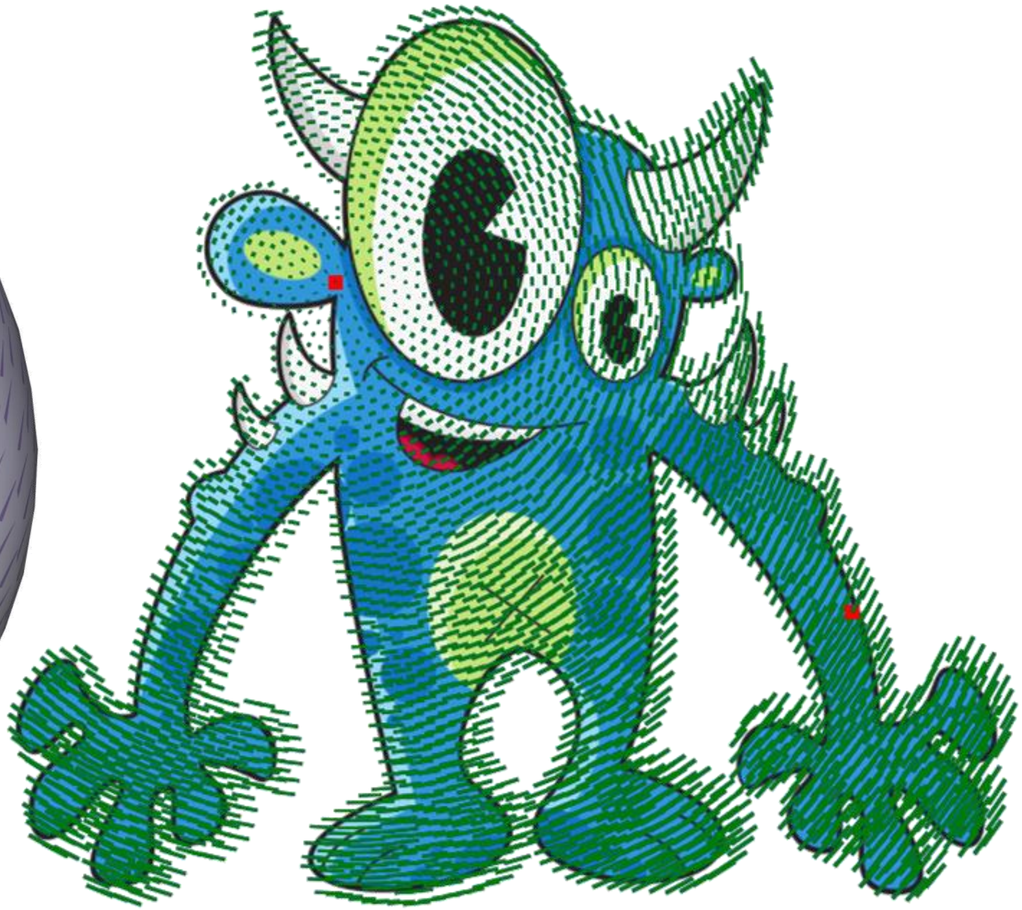
Continuous Symmetries



http://www.bradleycorp.com/image/985/9184b_highres.jpg

Killing Vector Fields





Velocity of Isometric Deformation

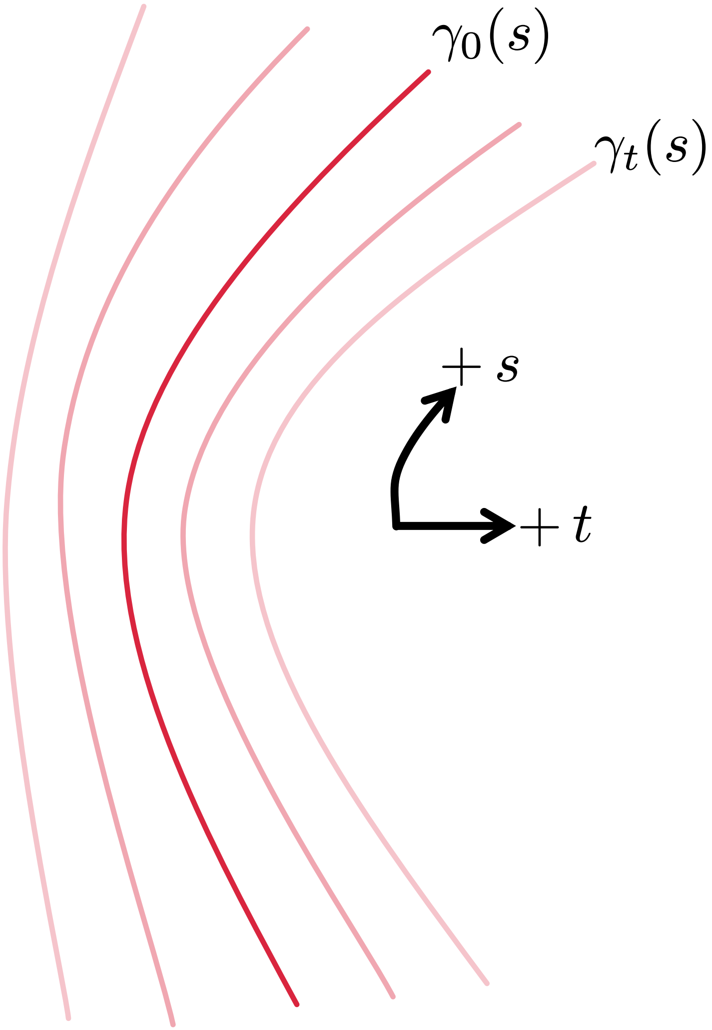
Wilhelm Killing

1847-1923

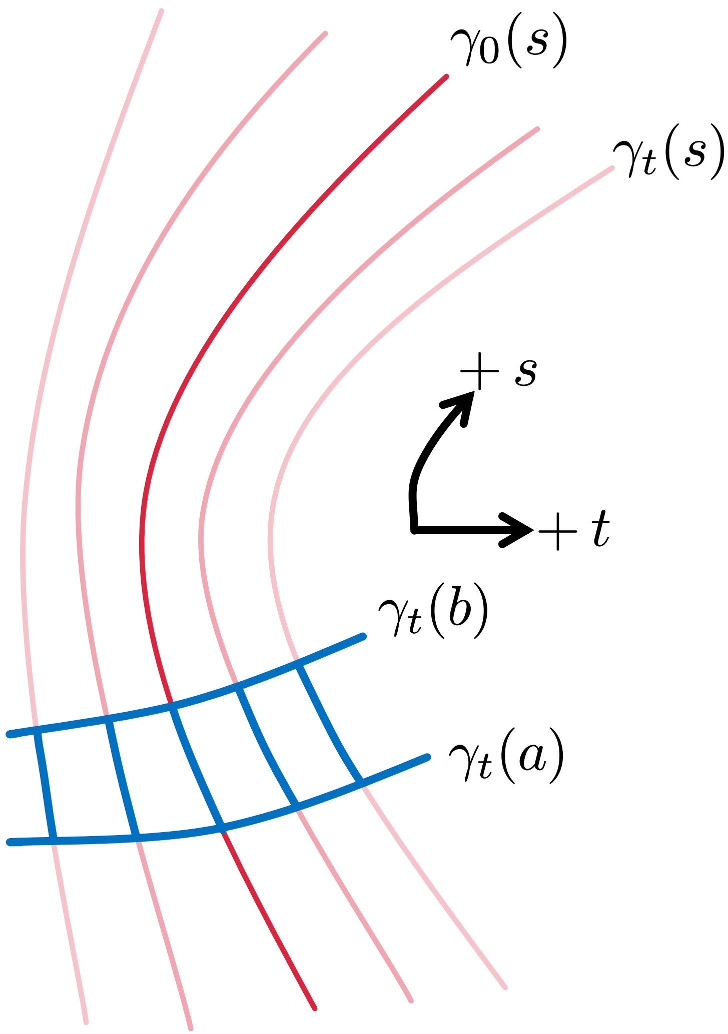
Germany

PRO TIP **Search "Killing *vector* fields"**

Families of Curves



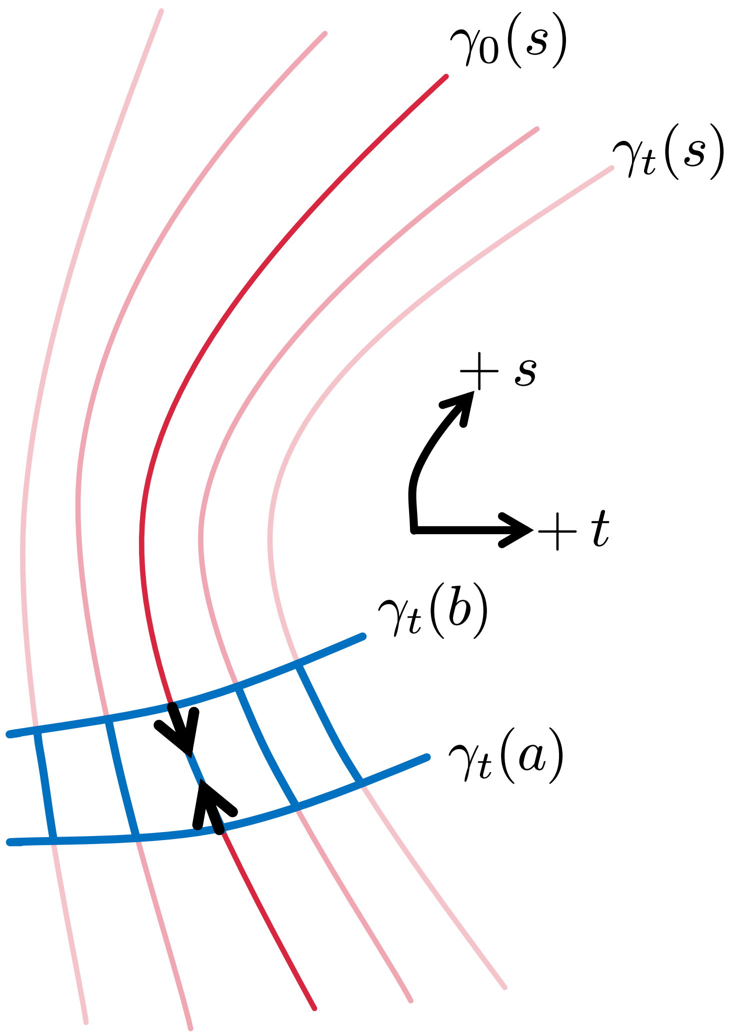
Families of Curves



Arc length constant in [a,b]:

$$\int_a^b \|\gamma_0'(s)\|ds = \int_a^b \|\gamma_t'(s)\|ds$$

Families of Curves



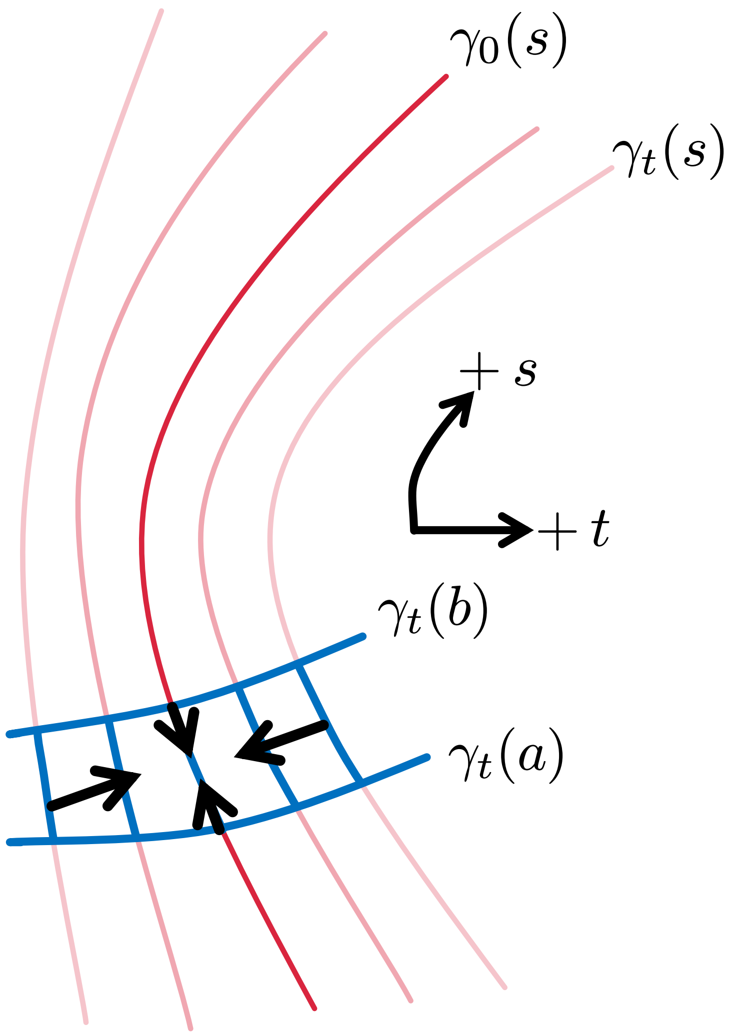
Arc length constant in [*a*,*b*]:

$$\int_a^b \|\gamma_0'(s)\| ds = \int_a^b \|\gamma_t'(s)\| ds$$

Holds for all [*a*,*b*]:

$$\|\gamma_0'(s)\| = \|\gamma_t'(s)\|$$

Families of Curves



Arc length constant in [*a*,*b*]:

$$\int_a^b \|\gamma_0'(s)\| ds = \int_a^b \|\gamma_t'(s)\| ds$$

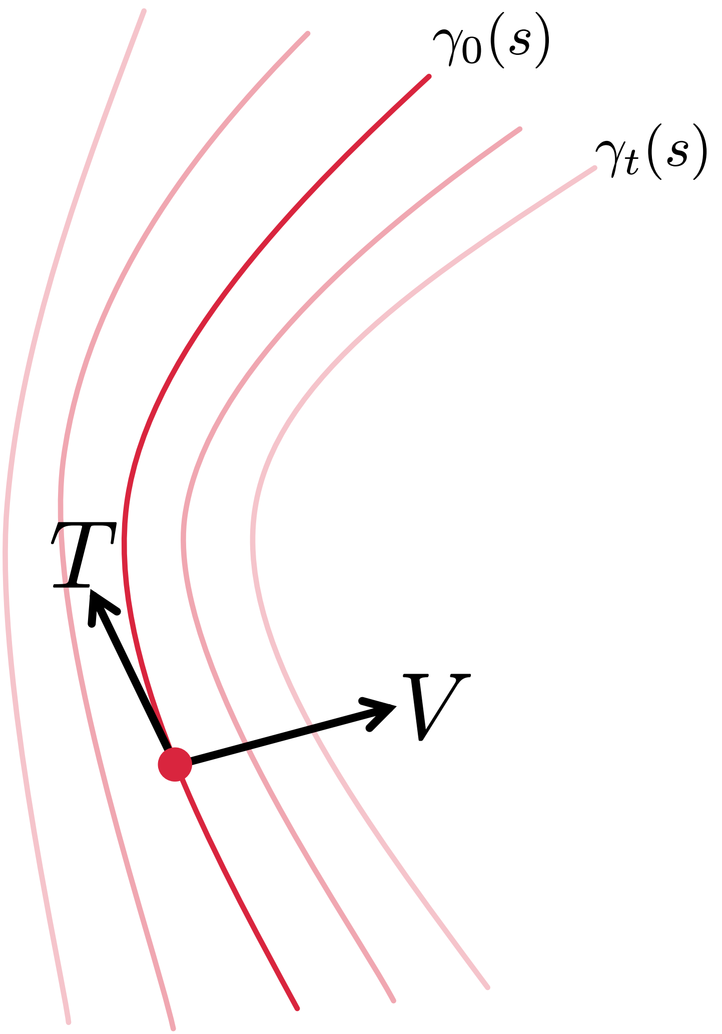
Holds for all [*a*,*b*]:

$$\|\gamma_0'(s)\| = \|\gamma_t'(s)\|$$

Differentiate in *t*:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\|\gamma_t'(s)\| \\ &= \frac{1}{\|\gamma_t'(s)\|}\left\langle \gamma_t'(s), \frac{\partial}{\partial t}\gamma_t'(s)\right\rangle \\ &= \left\langle T, \frac{\partial}{\partial t}\gamma_t'(s)\right\rangle \end{aligned}$$

First-Order Isometry Condition



$$0 \equiv \langle T, \frac{\partial}{\partial t}\gamma_t'(s)\rangle$$

$$\Updownarrow$$

$$\boxed{0 \equiv \langle T, D_T V\rangle}$$

KVF Condition

$$0 \equiv \langle T, D_T V \rangle$$

for all tangent vector fields T

Fix length of all curves

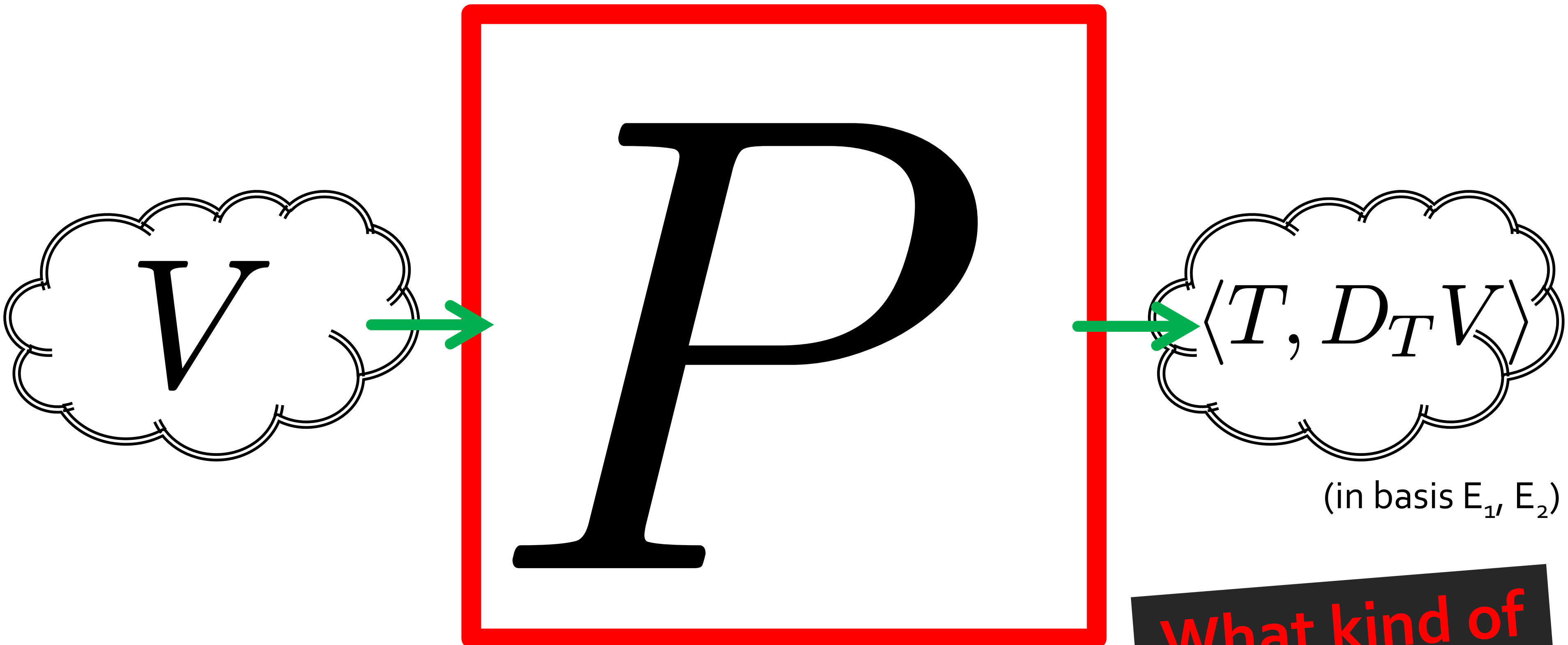
Relationships with KVFs

$$0 \equiv \langle T, D_T V \rangle$$ **EKVF**

Same tangential component

$$0 \equiv \langle T, \nabla_T V \rangle$$ **KVF**

Killing Operator

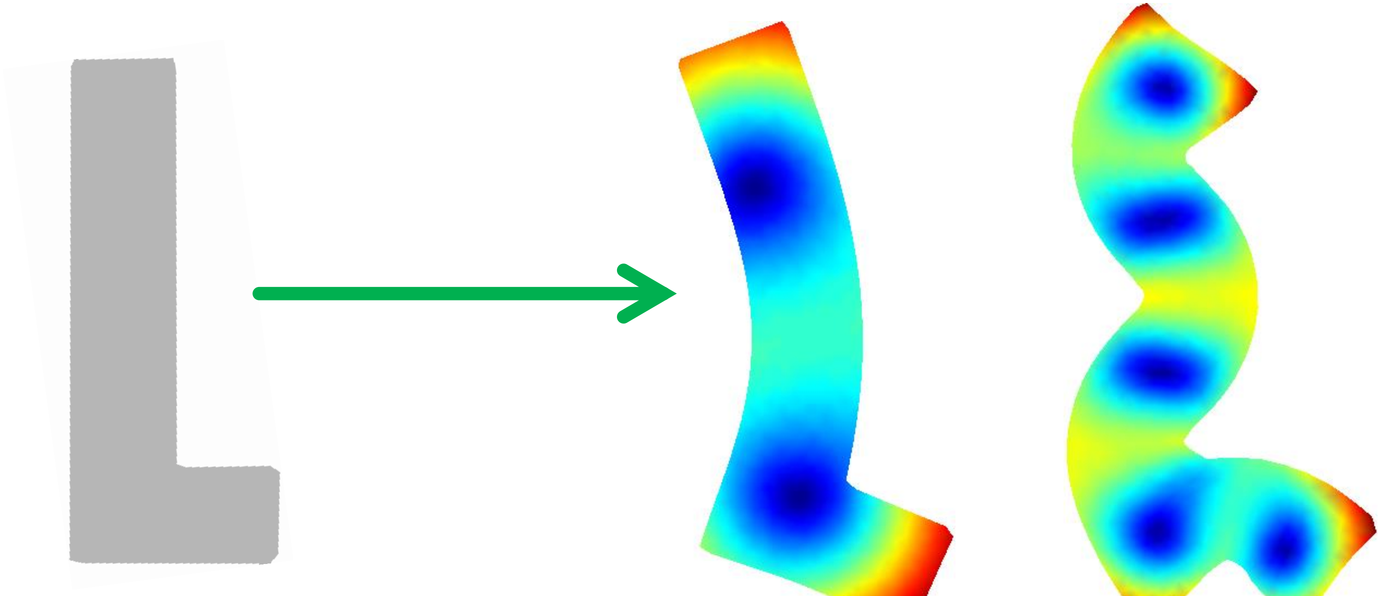


(in basis E_1, E_2)

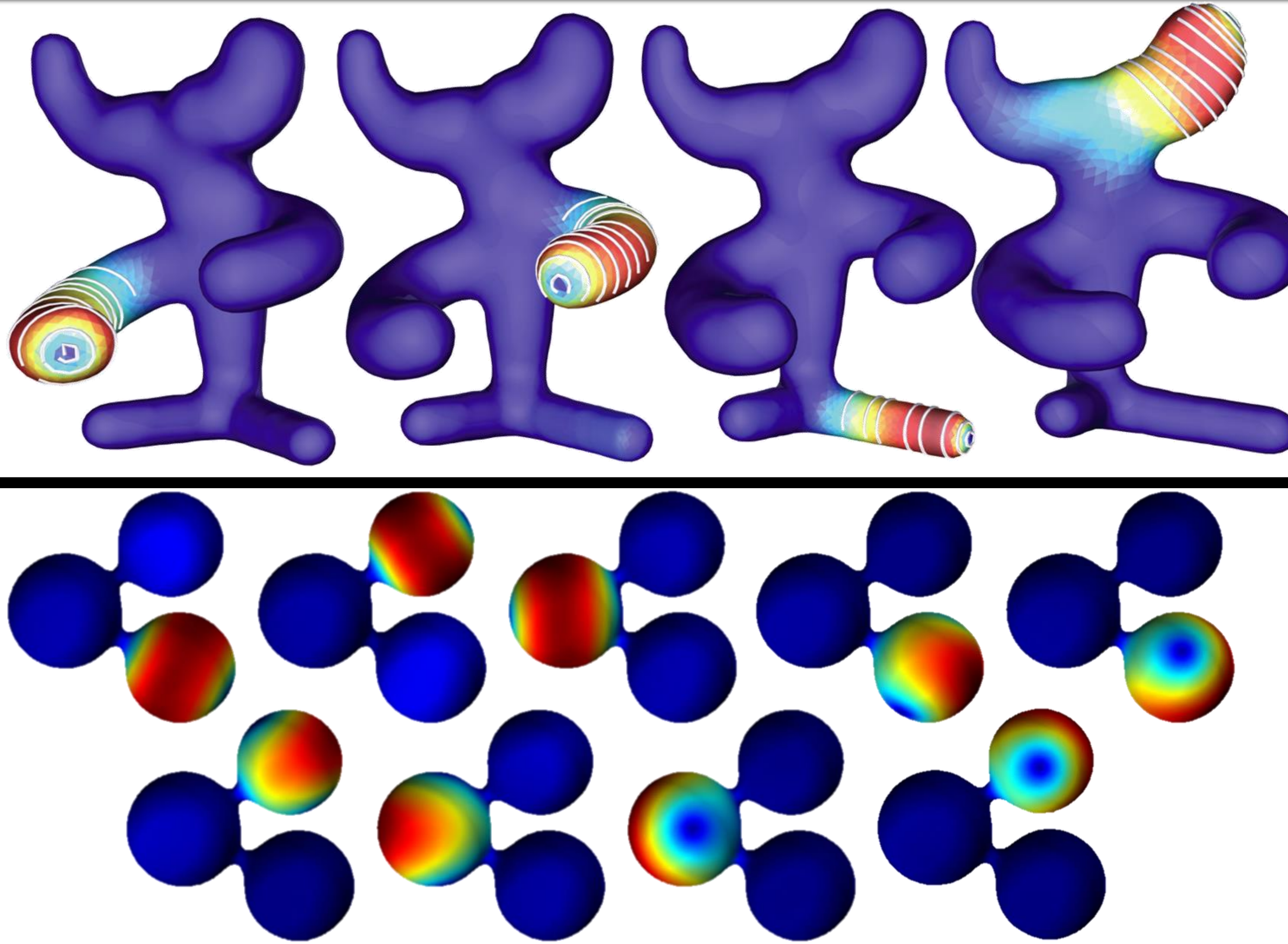
What kind of object is this?

Killing Energy

$$\int_{\Sigma} \|PV\|^2$$

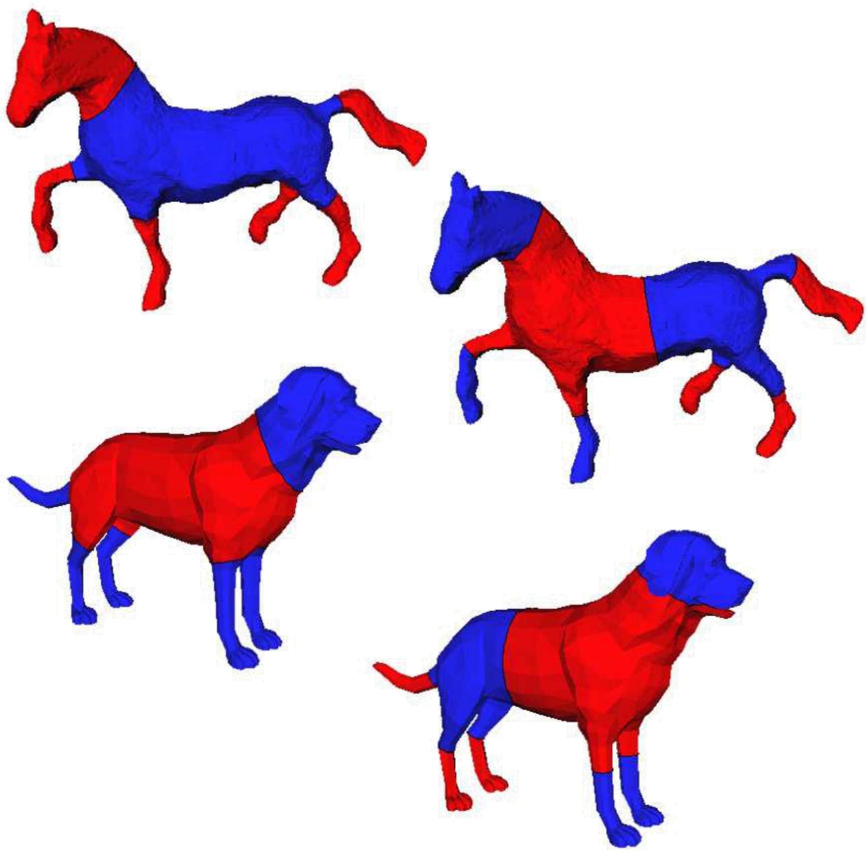


KVF Operator Eigenfunctions

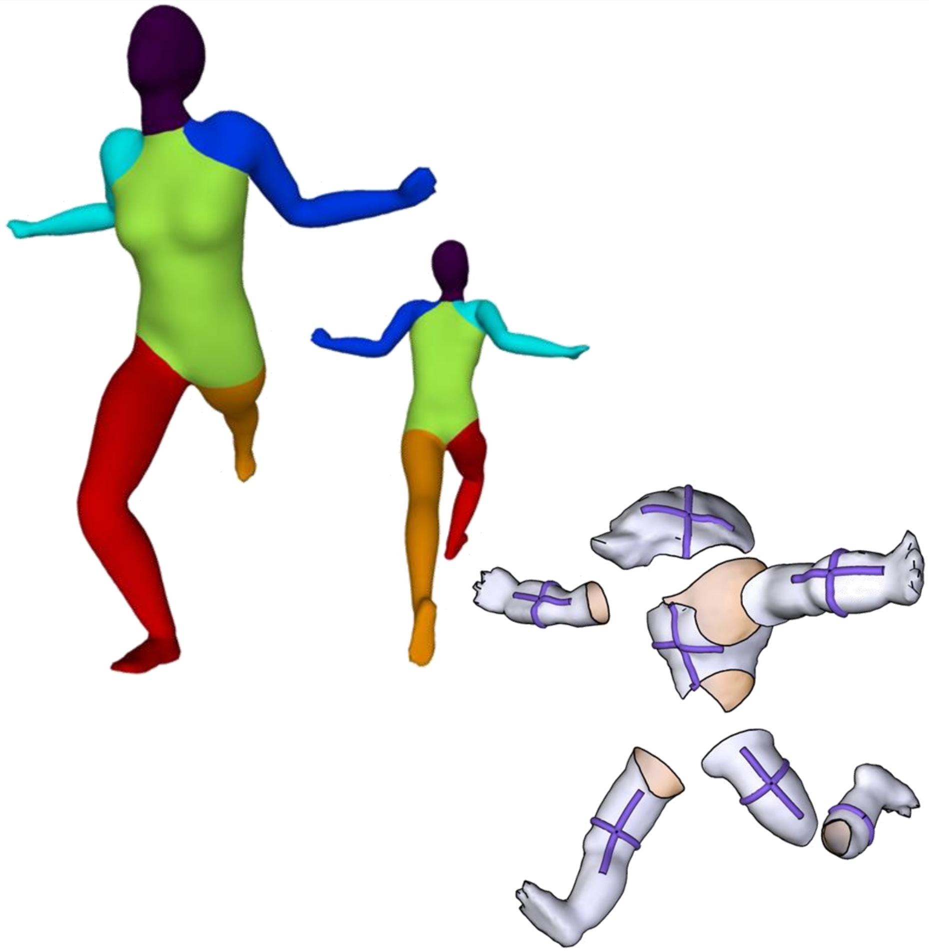


http://www.stanford.edu/~justso1/assets/intrinsic_part_discovery.pdf

Application: Segmentation



Laplacian nodal sets

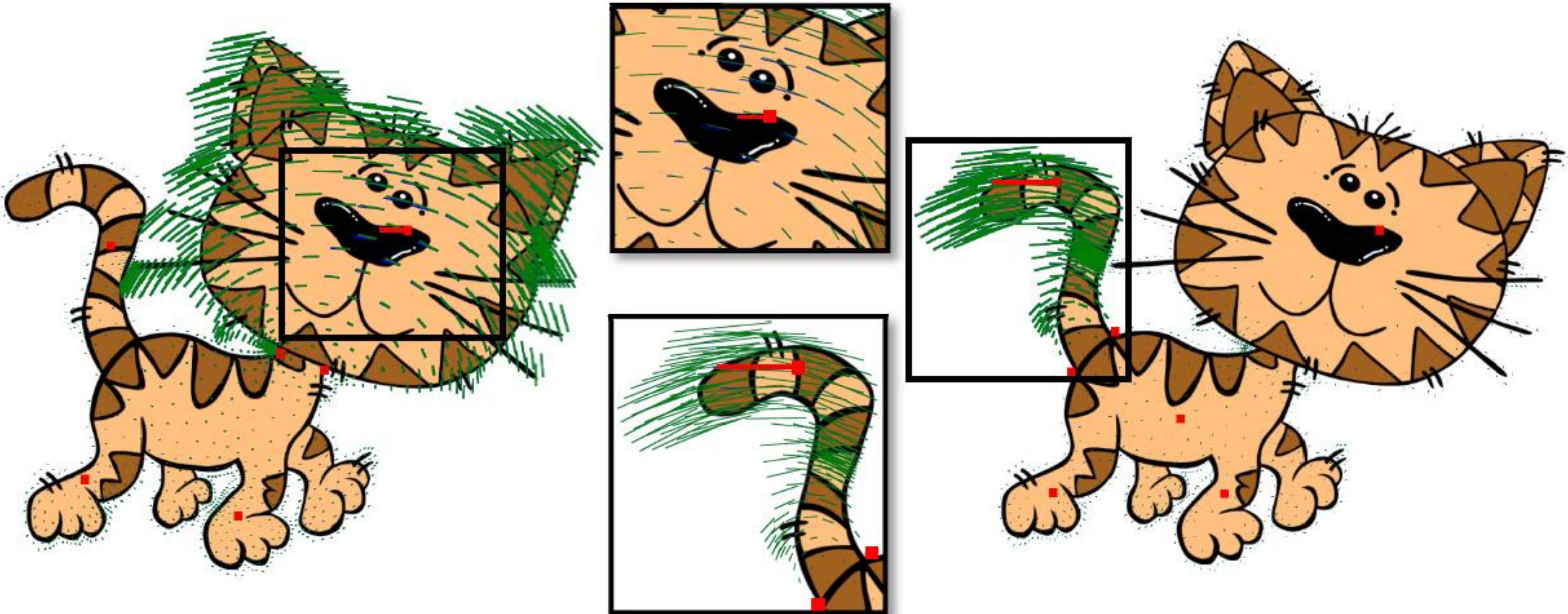


KVF eigenfunctions

reuter.mit.edu/papers/reuter-smi09.pdf http://www.stanford.edu/~justso1/assets/intrinsic_part_discovery.pdf

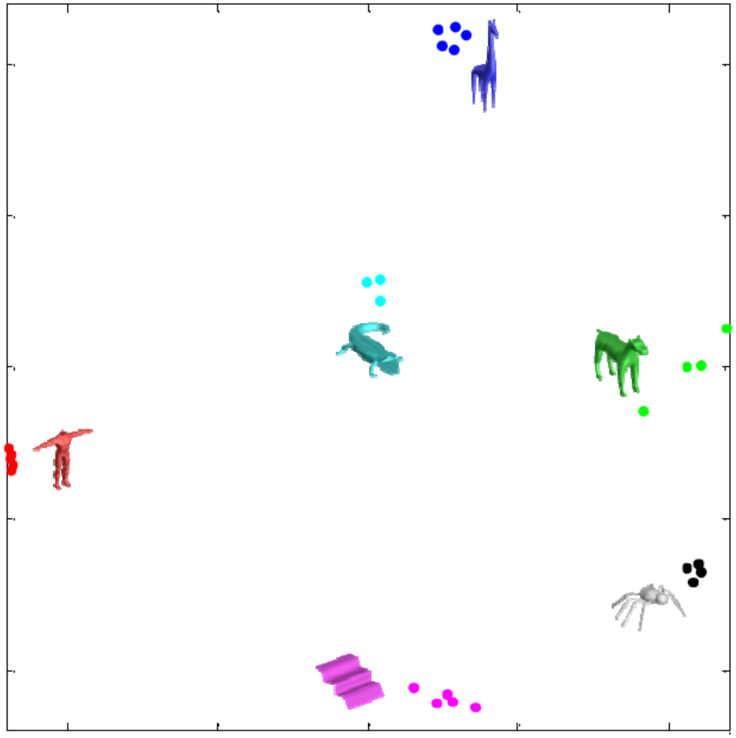
Application: 2D Deformation

$$\min_{V:\ \to\mathbb{R}^2} \left(\lambda \sum_{p_i \in C} \|V(p_i) - \vec{v}_i\| + \int \|PV\|^2 \right)$$



http://www.stanford.edu/~justso1/assets/kvf_deformation.pdf

Application: Shape Distance



Hausdorff

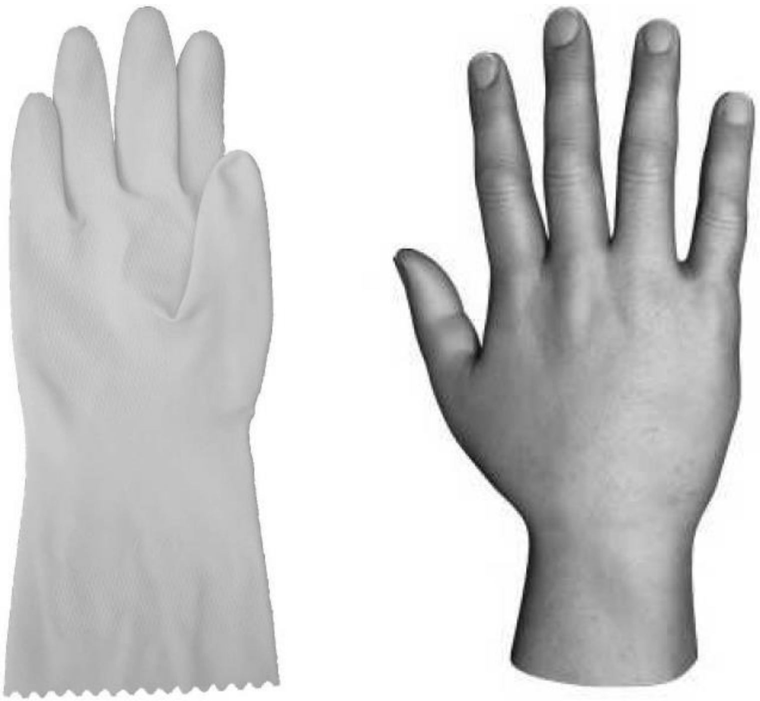
$$d_H(X,Y) \equiv \max\{\sup_{x\in X} \inf_{y\in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y)\}$$

Gromov-Hausdorff

$$d_{GH}(X,Y) \equiv \inf_{I,J \text{ isometries}} d_H(I(X), J(Y))$$

http://visl.technion.ac.il/bron/publications/BroBroKimSIAMo6.pdf

Beware



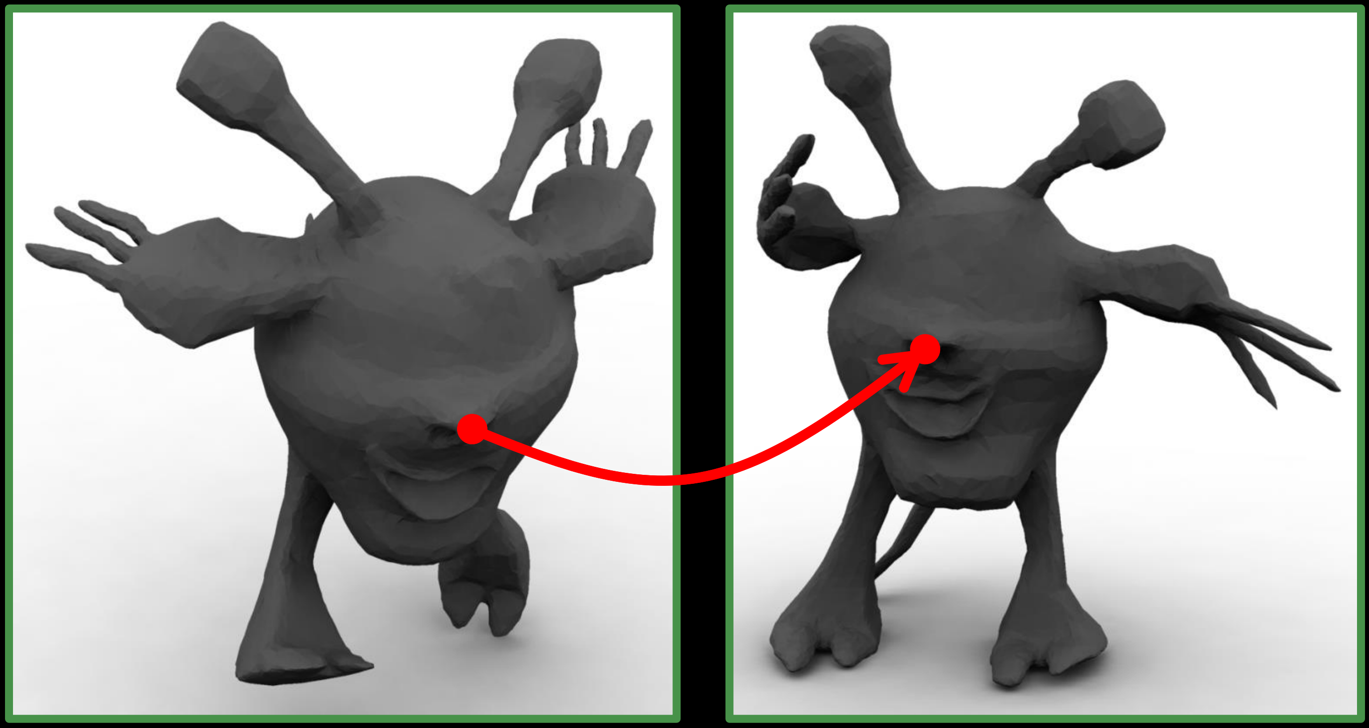
http://www.cs.technion.ac.il/~mbron/publications_conference.html

Not the same.

Tons of Applications

- Segmentation
- Symmetry detection
- Global shape description
- Retrieval
- Recognition
- Feature extraction
- Alignment
- …

Only scratched the surface!



Isometry Invariance and Spectral Techniques

CS 468, Spring 2013
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher