CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 11: Derivatives

Derivatives in Euclidean space.

- Derivatives of f a scalar function in the direction of a vector.
- Derivatives of vector fields viewed as n-tuples of scalar functions along curves: $[\bar{\nabla}_X Y]_p := [XY]_p = \sum_i \lim_{\varepsilon \to 0} \varepsilon^{-1}(Y^i(p + \varepsilon X) - Y^i(p))E_i$.
- Four properties of the vector field derivative: the C^∞-linearity in the X-slot, the $\mathbb{R}$-linearity in the Y-slot, the Leibniz rule in the Y-slot, the metric compatibility property.
- Derivatives of vector fields when moving frames are involved. Christoffel symbols.

Derivatives of objects defined on a surface.

- Derivatives of a scalar function in the direction of a vector on a surface.
- Derivatives of a vector field: there are problems...
  - The *geometric definition* of differentiation (limit of finite differences) fails since the derivative of a vector field on the surface may not be tangent to the surface. What's the alternative — how can we compare vectors in different tangent spaces?
  - Using a parametrization involves moving frames (for a parametrization $\phi : \Omega \to \mathbb{R}^3$ the moving frame is $E_i := D\phi(\frac{\partial}{\partial u^i})$). But still derivatives may not be tangent to the surface.
- Definition of the covariant derivative as $\nabla_X Y := [\bar{\nabla}_X Y]^{\|}$. Given vector fields X, Y then $(\nabla_X Y)_p$ depends on X_p and Y along a curve passing tangentially through X_p only. Alternate notation $\frac{DY}{dt}$ for $\nabla_X Y$ when $c : I \to \mathbb{R}$ is a curve and $X := \dot{c}$ and Y is defined along c.
- Thus we have a relationship to second fundamental form $\bar{\nabla}_X Y = A(X, Y)N + \nabla_X Y$.
- The four properties of the vector field derivative revisited.
- The Lie bracket of two vector fields: the Lie bracket of two coordinate vector fields vanishes; therefore it's a tangent vector field if the individual vector fields are. The torsion-free property.

Parallel transport.

- Definition of parallel transport and its properties. The tangent vector of a geodesic is parallel transported along itself. Geodesic equations.
- A new geometric definition of differentiation — comparison of vectors in different tangent spaces using parallel transport.

The fundamental lemma of Riemannian geometry.

- The Christoffel symbols
- Fundamental Lemma of Riemannian Geometry. Covariant derivatives are intrinsic.

Gradient, divergence and Laplacian.

- The gradient vector.
- Definition of divergence using an orthonormal frame. Independence of frame.
- The divergence theorem.
- The Laplacian. Harmonic functions, etc.. Integration-by-parts formulas.