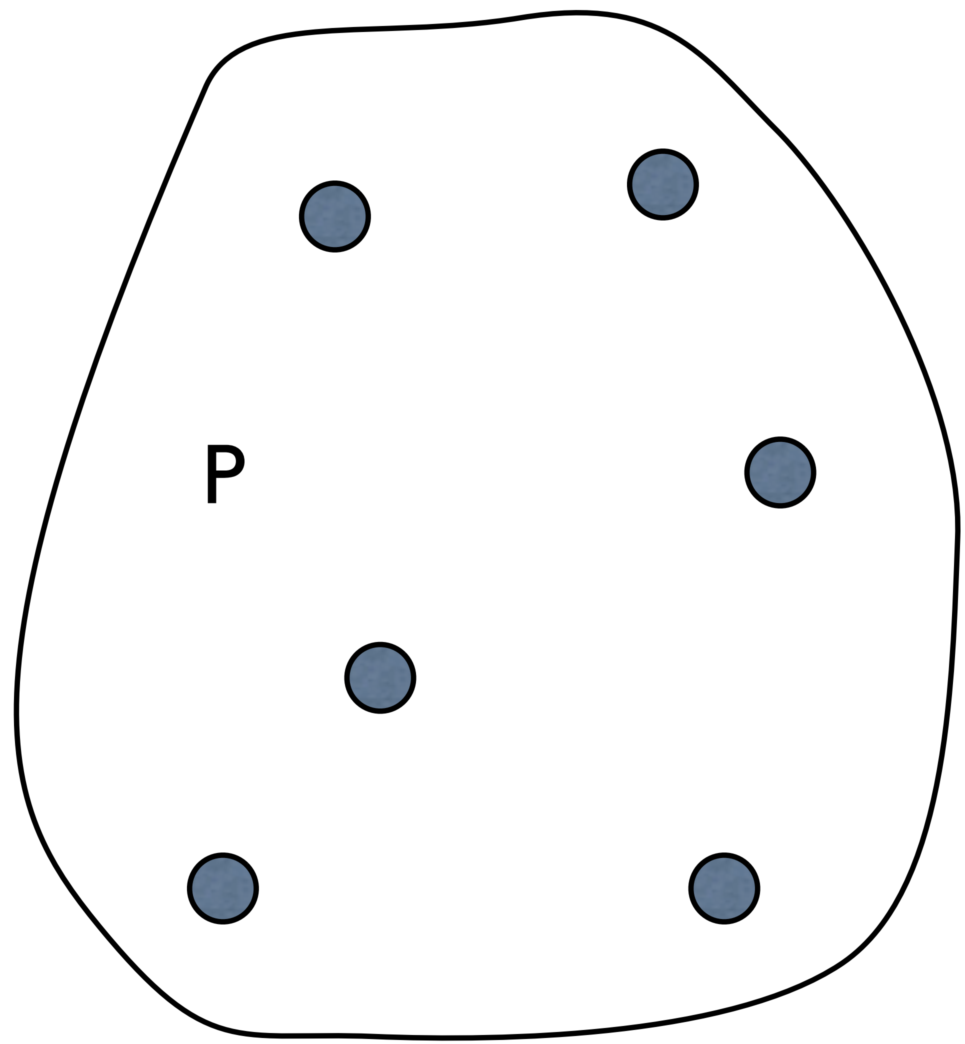
# Approximate Nearest Neighbor via Point-Location among Balls

# Outline

- Problem and Motivation
- Related Work
- Background Techniques
- Method of Har-Peled (in notes)

# Problem

- P is a set of points in a metric space.
- Build a data structure to efficiently search ANN



# Motivation

- Nearest Neighbor Search has lots of applications.
- Curse of dimensionality

  - Voronoi diagram method exponential in dimension.
- Settle for approximate answers.

# Related Work

- Indyk and Motwani
- Approximate Nearest Neighbors: Towards Removing the Curse of Dimensionality
- Reduced ANN to Approximate Point-Location among Equal Balls.
- Polynomial construction time.
- Sublinear query time.

# Related Work

- Har-Peled
- A Replacement for Voronoi Diagrams of Near Linear Size
- Simplified and improved Indyk-Motwani reduction.

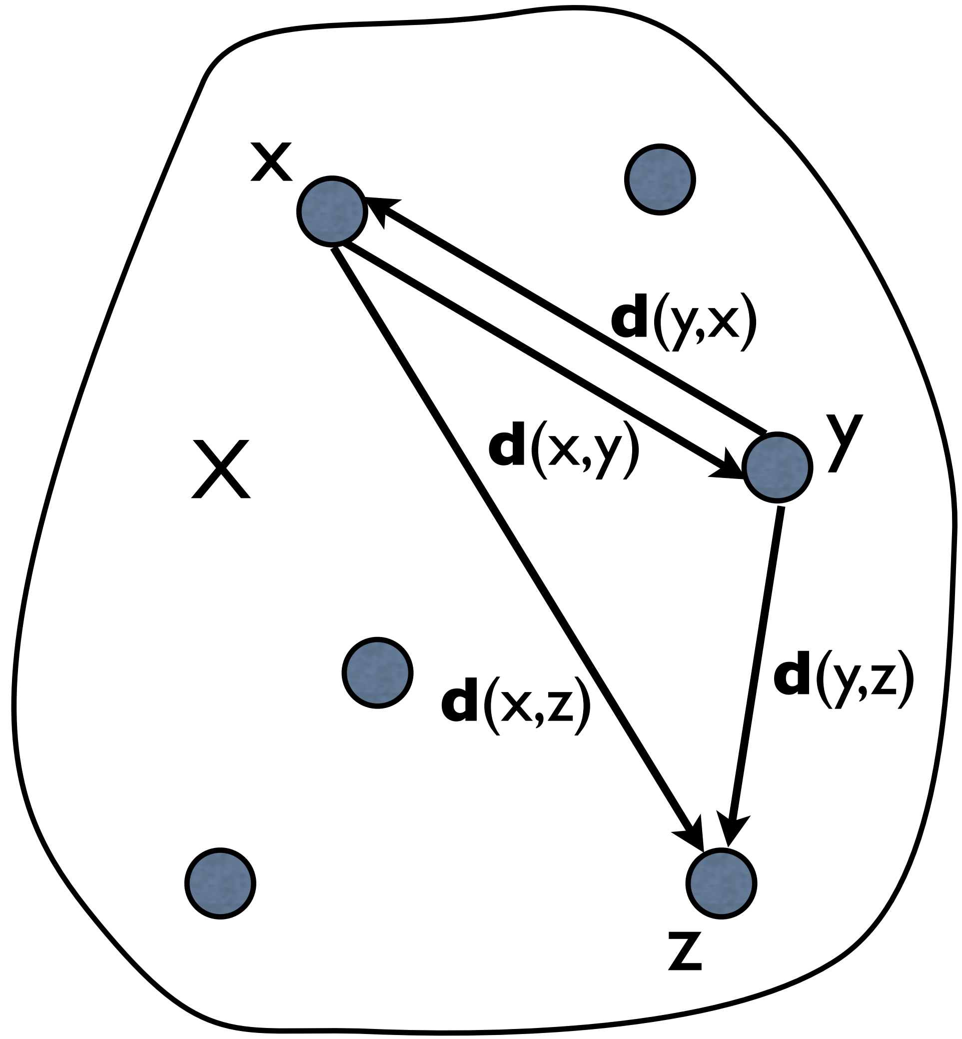
  - Better construction and query time.

# Related Work

- Sabharwal, Sharma and Sen
- Nearest Neighbors Search using Point Location in Balls with applications to approximate Voronoi Decompositions.
- Improved number of balls by a logarithmic factor.
- Also a complex construction which only requires O(n) balls.

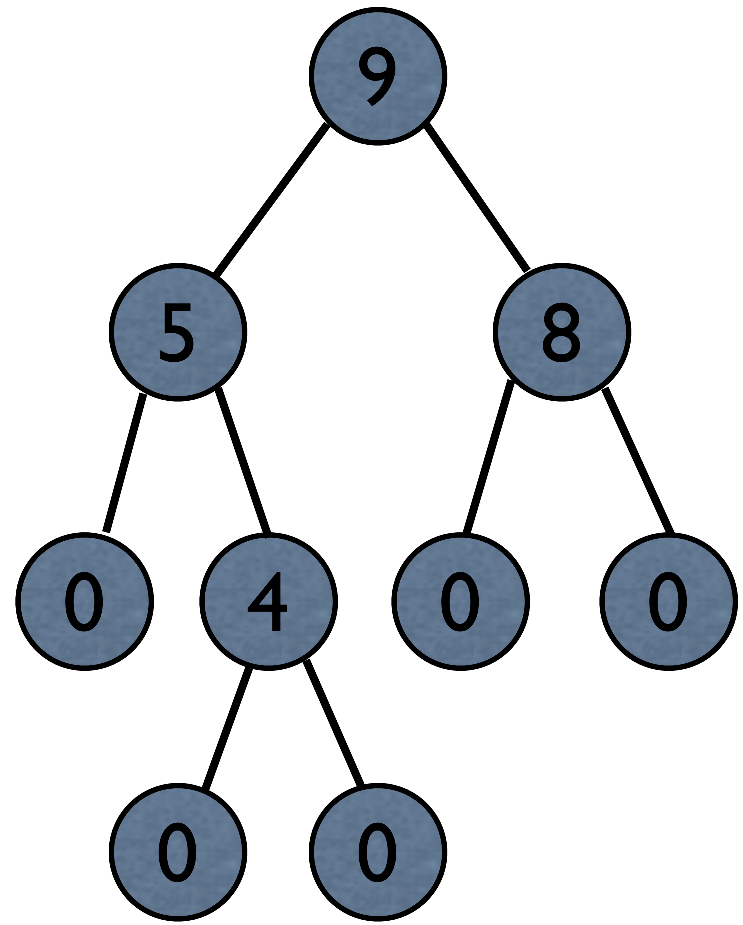
# Metric Spaces

- Pair (X,**d**)
- **d**: X × X → [0,∞)
- **d**(x,y) = 0 iff x = y
- **d**(x,y) = **d**(y,x)
- **d**(x,y) + d(y,z) ≥ d(x,z)



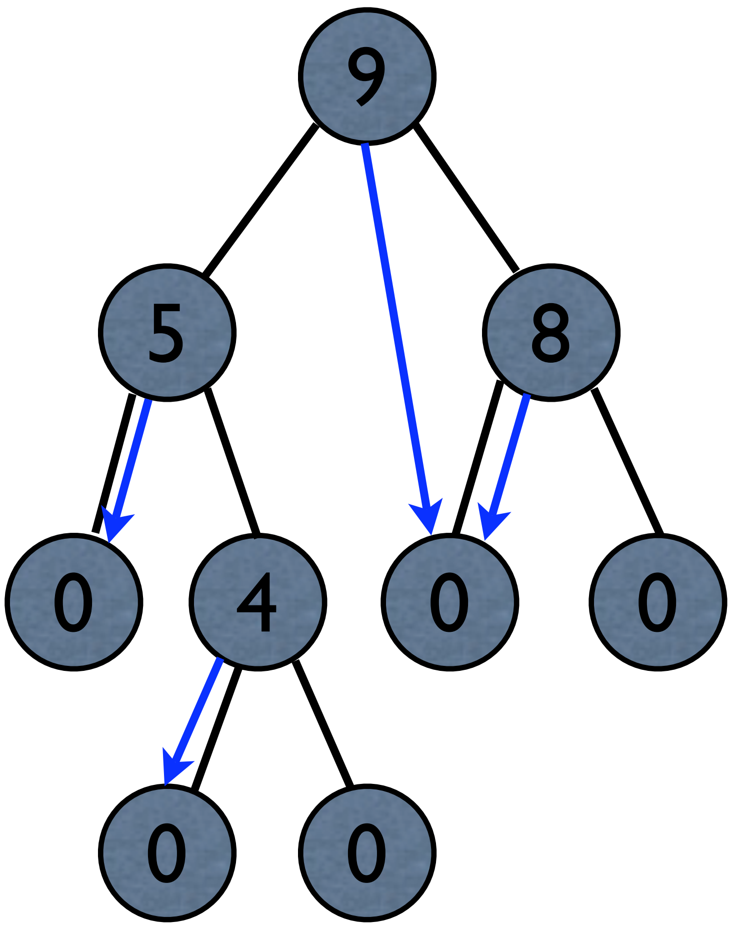
# Hierarchically well-Separated Tree (HST)

- Each vertex u has a label $\Delta_u \geq 0$.
- $\Delta_u = 0$ iff u is a leaf.
- If a vertex u is a child of a vertex v, then $\Delta_u \leq \Delta_v$.
- Distance between two leaves u,v is defined as $\Delta_{lca(u,v)}$ where lca is the least common ancestor.



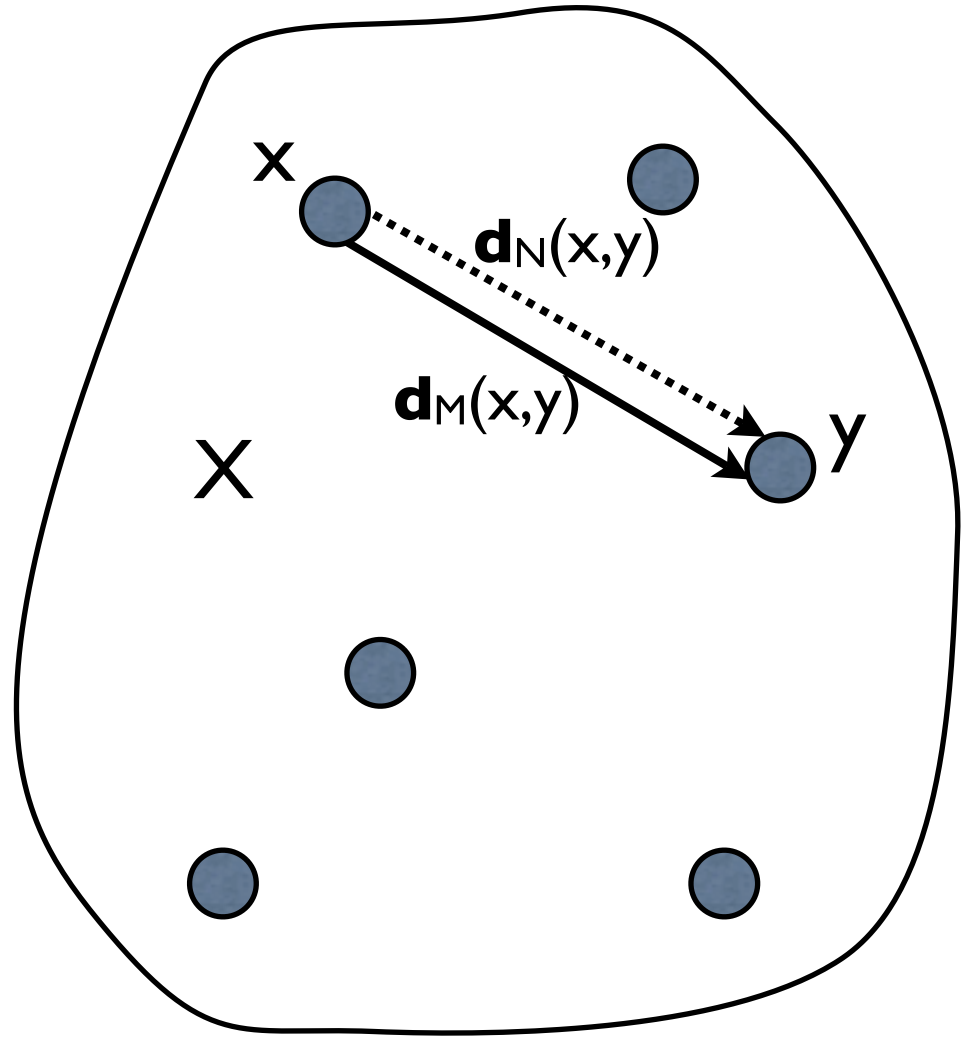
# Hierarchically well-Separated Tree (HST)

- Each vertex u has a representative descendant leaf $rep_u$.
- $rep_u \in \{rep_v \mid v \text{ is a child of } u\}$.
- If u is a leaf, then $rep_u = u$.

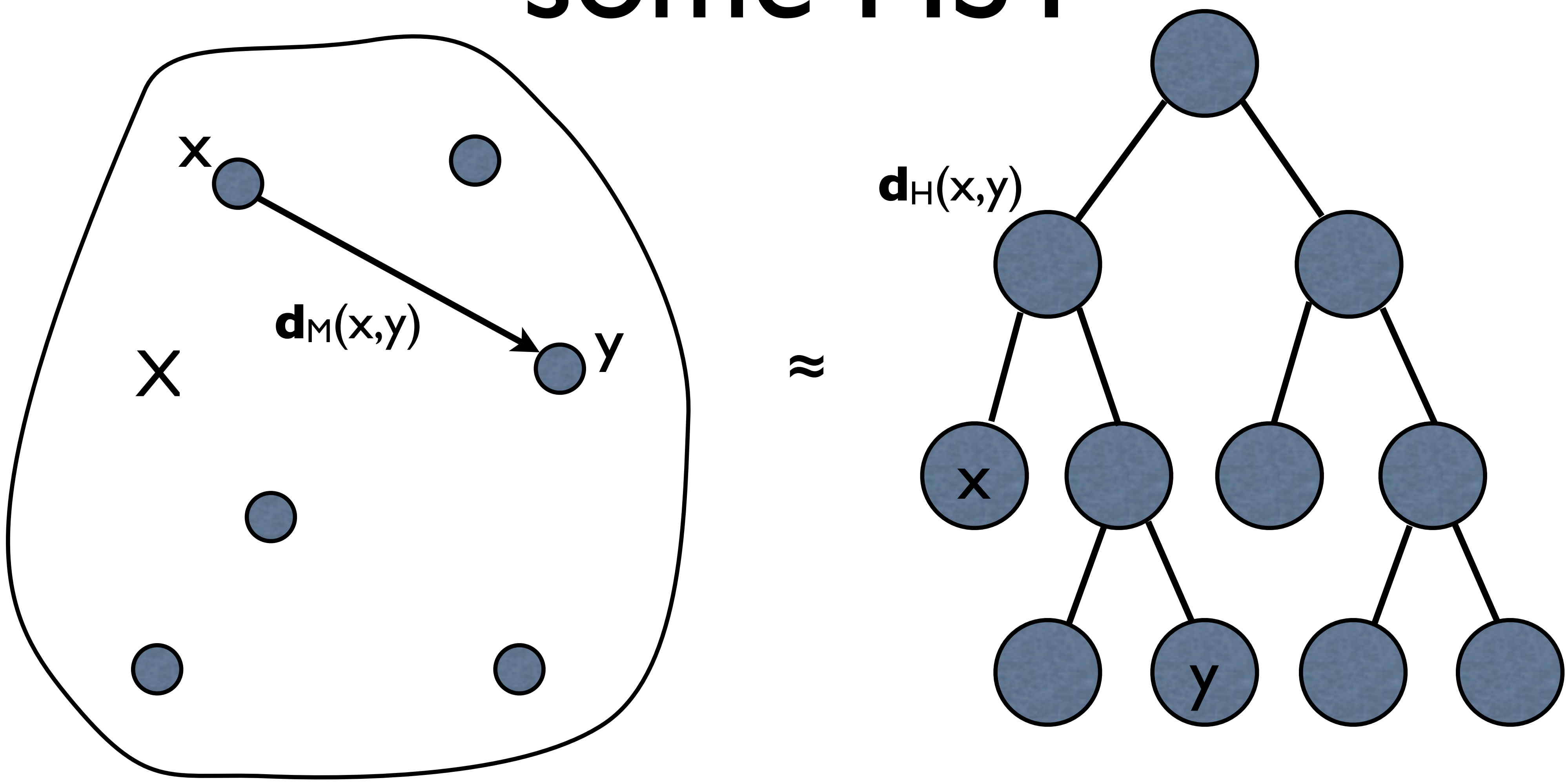


# Metric t-approximation

- A metric N t-approximates a metric M, if they are on the same set of points, and $\mathbf{d}_M(x,y) \le \mathbf{d}_N(x,y) \le t\mathbf{d}_M(x,y)$ for any points x,y.



# Any n-point metric is 2(n-1)-approximated by some HST

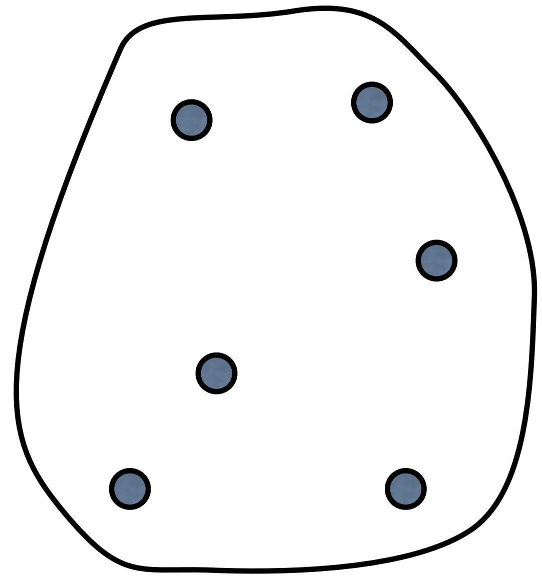


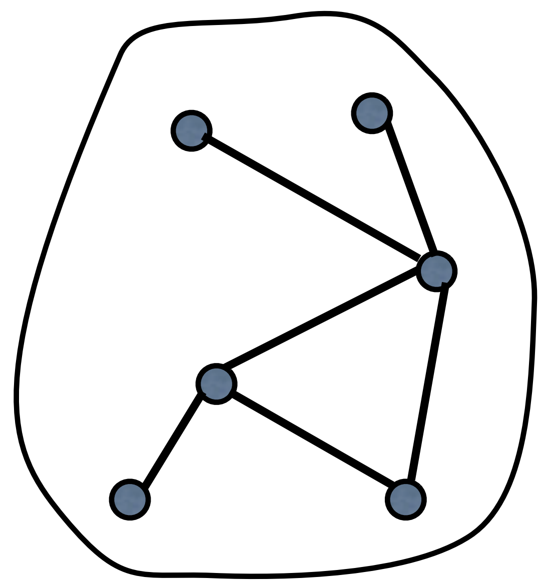
# First Step: Compute a 2-spanner

- Given a metric space M, a 2-spanner is a weighted graph G whose vertices are the point of M and whose shortest path metric 2-approximates M.

- $\mathbf{d}_M(x,y) \leq \mathbf{d}_G(x,y) \leq 2\mathbf{d}_M(x,y)$ for all x,y.

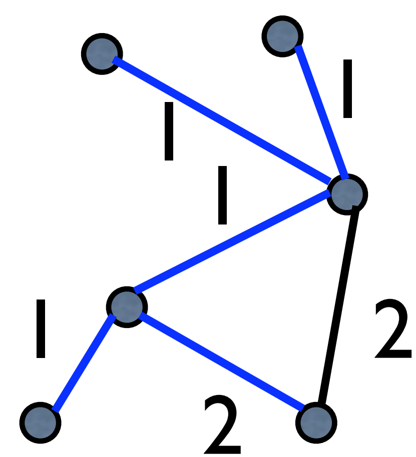
- Can be computed in O(nlogn) time — Details in Chapter 4.





# Construct a HST which (n-1)-approximates the 2-spanner

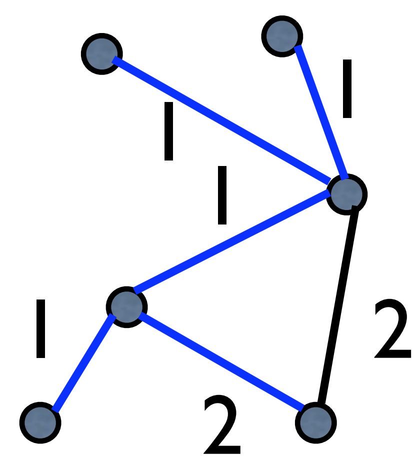
- Compute the minimum spanning tree of G, the 2-spanner



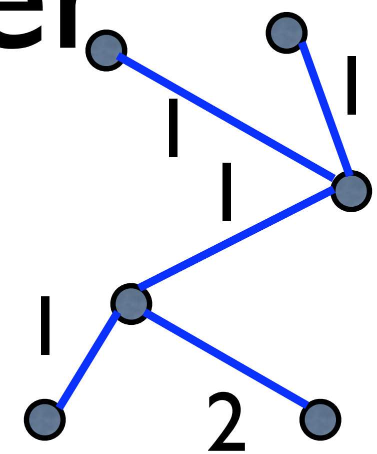
# Construct a HST which (n-1)-approximates the 2-spanner

- Construct the HST using a variation of Kruskal's algorithm

- Order the edges in non-decreasing order.

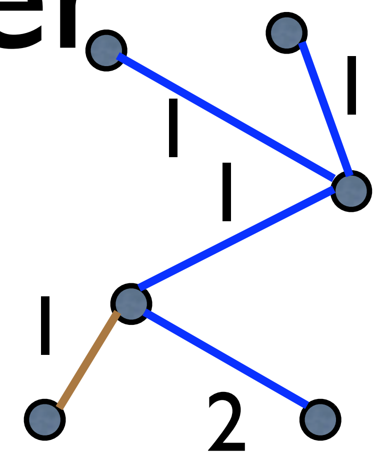


# Construct a HST which (n-1)-approximates the 2-spanner

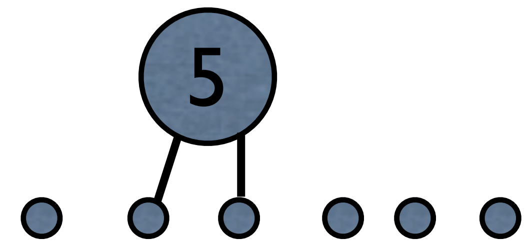



- Start with n 1-element HSTs.

# Construct a HST which (n-1)-approximates the 2-spanner




- Add the edges one by one, and merge corresponding HSTs by adding a parent node with Δ label equal to (n-1) times the edge's weight.



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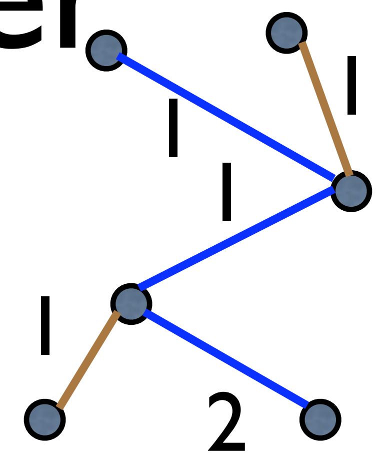


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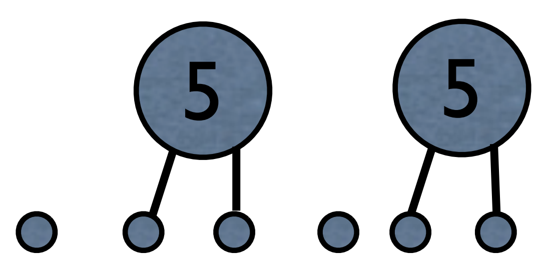

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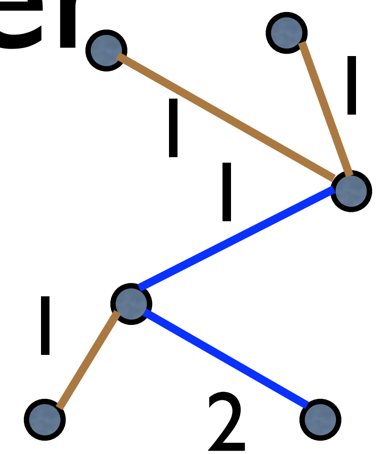


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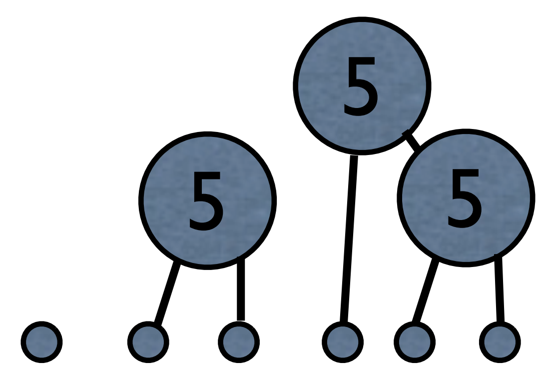

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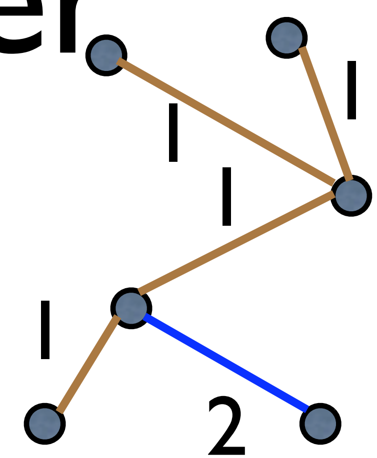


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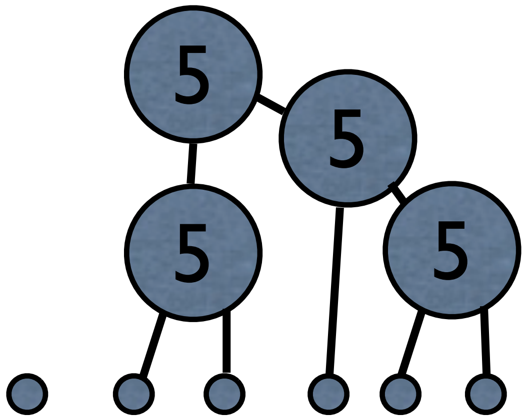
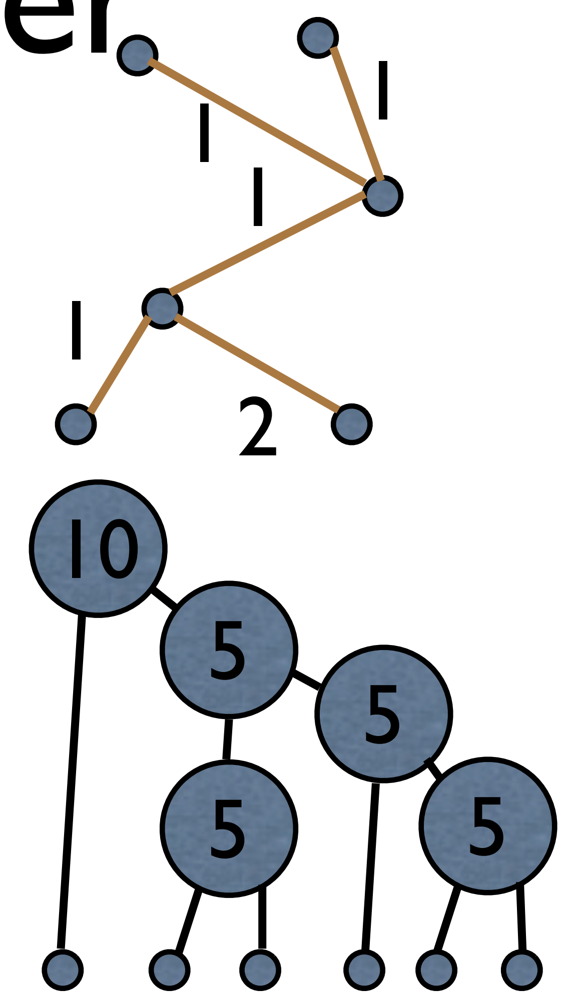


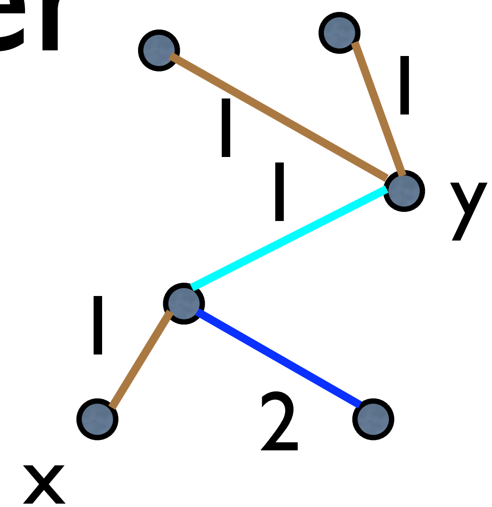
# Construct a HST which (n-1)-approximates the 2-spanner

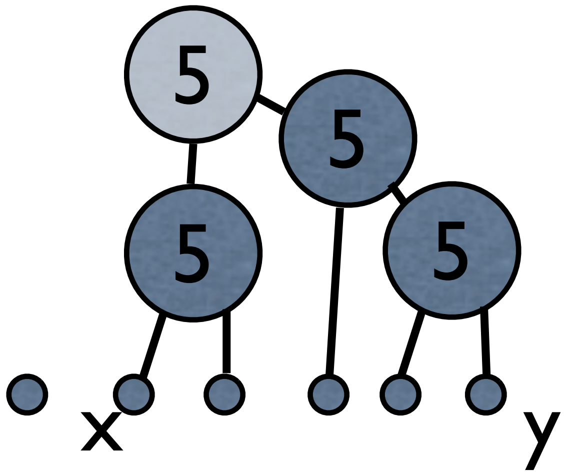
- Add the edges one by one, and merge corresponding HSTs by adding a parent node with $\Delta$ label equal to (n-1) times the edge's weight.



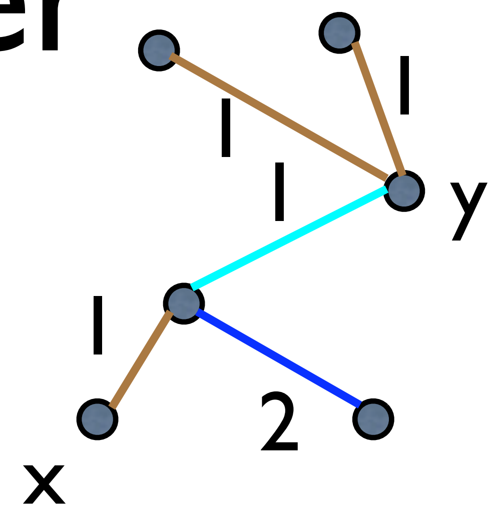
# The HST (n-1)-approximates the 2-spanner

- Consider vertices x and y in the graph and the first edge e that connects their respective connected components.



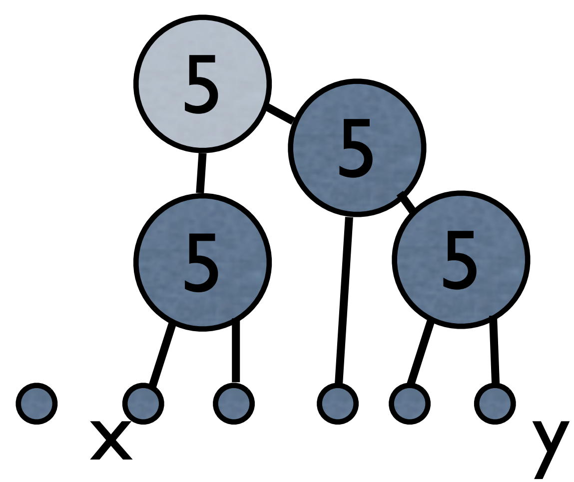
# The HST (n-1)-approximates the 2-spanner



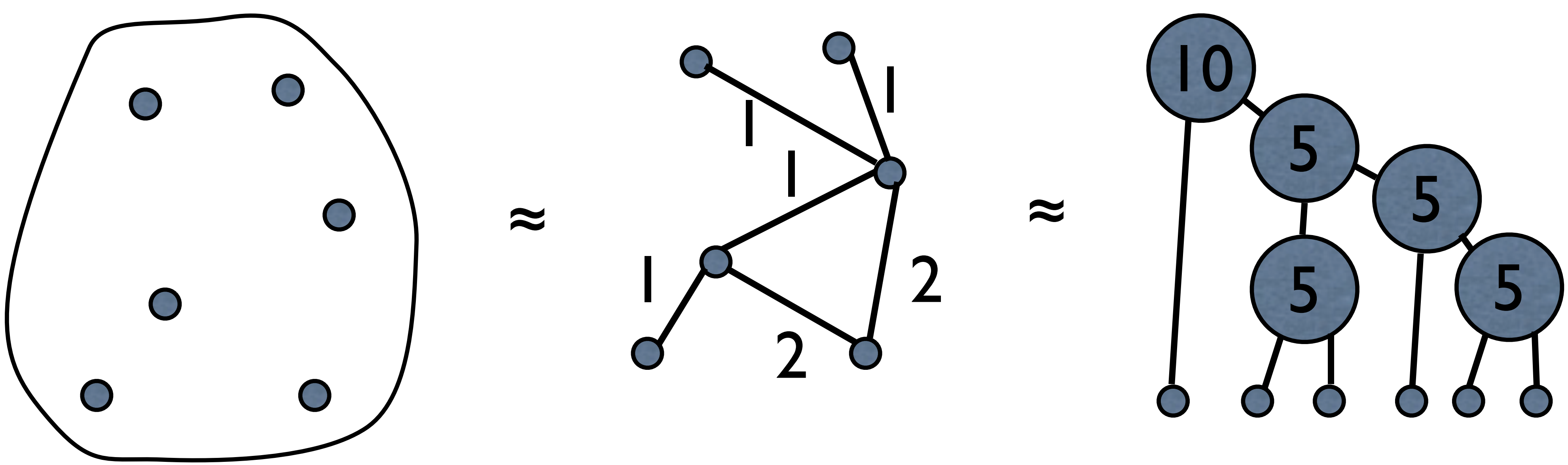

- Let C be the connected component containing x and y after e is added.

- $w(e) \leq \mathbf{d}_G(x,y) \leq (|C|-1)w(e) \leq (n-1)w(e) = \mathbf{d}_H(x,y)$

- $\mathbf{d}_G(x,y) \leq \mathbf{d}_H(x,y) \leq (n-1)\,\mathbf{d}_G(x,y)$



# Any n-point metric is 2(n-1)-approximated by some HST

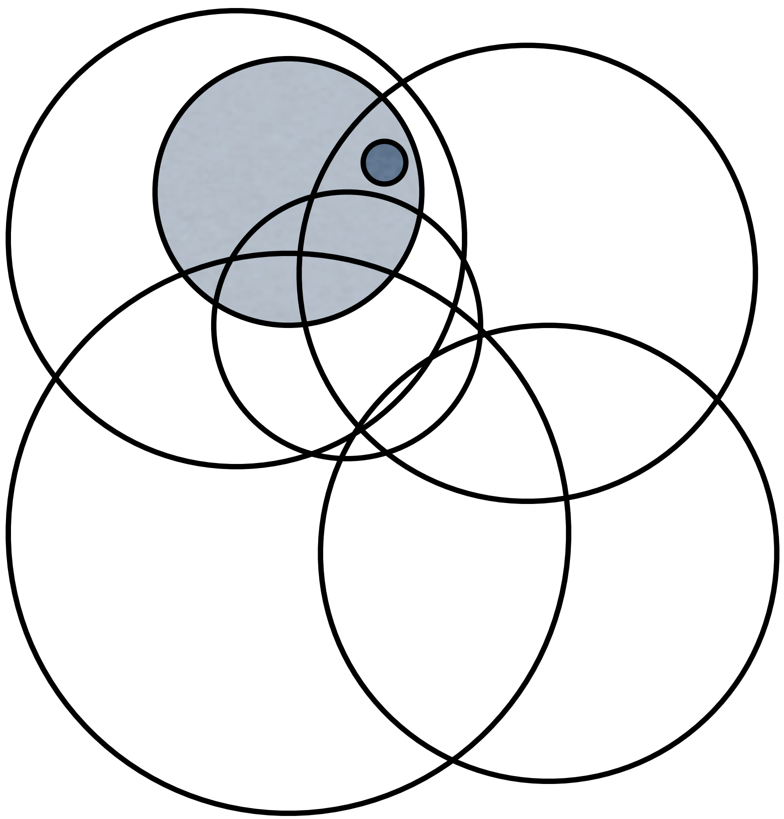


# Target Balls

- Let B be a set of balls such that the union of the balls in B contains the metric space M.

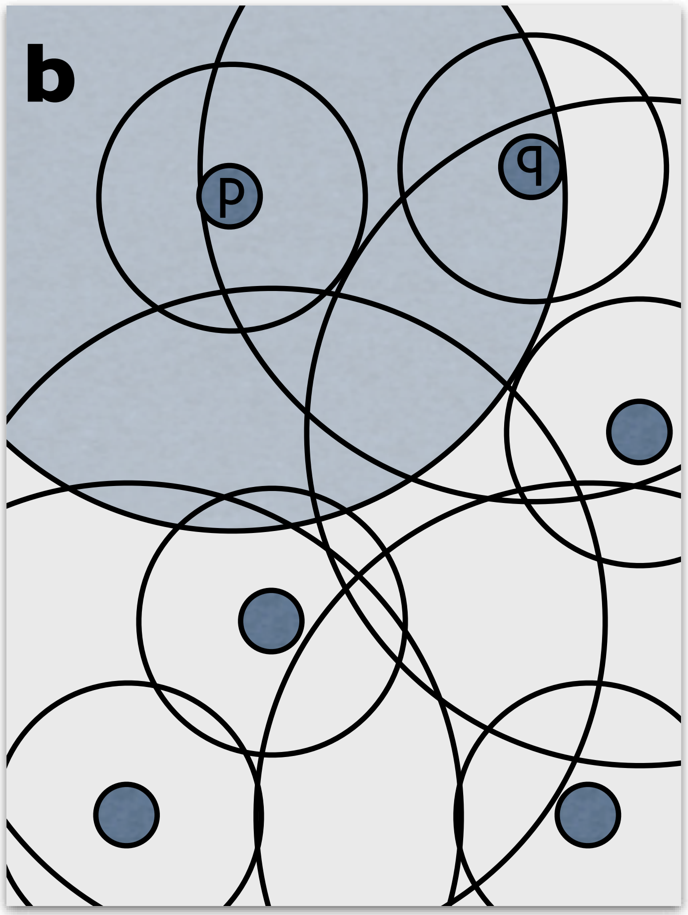
- For a point q in M, the target ball of q in B, denoted $\odot_B(q)$, is the smallest ball in B that contains q.

- We want to reduce ANN to target ball queries.



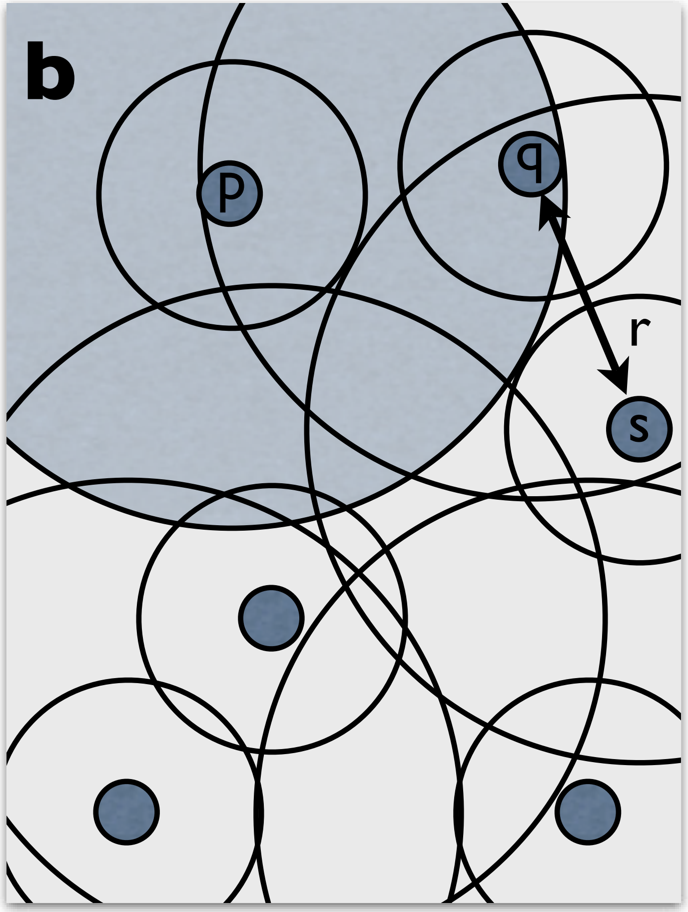
# A Trivial Result — Using Balls to Find ANN

- Let B(P,r) be the set of balls of radius r around each point p in P.
- Let B be the union of B(P, $(1+\epsilon)^i$) where i ranges from $-\infty$ to $\infty$.
- For a point q, let p be the center of **b** = $\odot_B(q)$. Then p is $(1+\epsilon)$-ANN to q.



# A Trivial Result — Using Balls to Find ANN

- Let s be the nearest neighbor to q in P.
- Let r = **d**(s,q).
- Fix i such that $(1+\epsilon)^i < r \leq (1+\epsilon)^{i+1}$
- Radius of **b** > $(1+\epsilon)^i$
- **d**(s,q) ≤ **d**(p,q) ≤ $(1+\epsilon)^{i+1}$ ≤ $(1+\epsilon)$**d**(s,q)

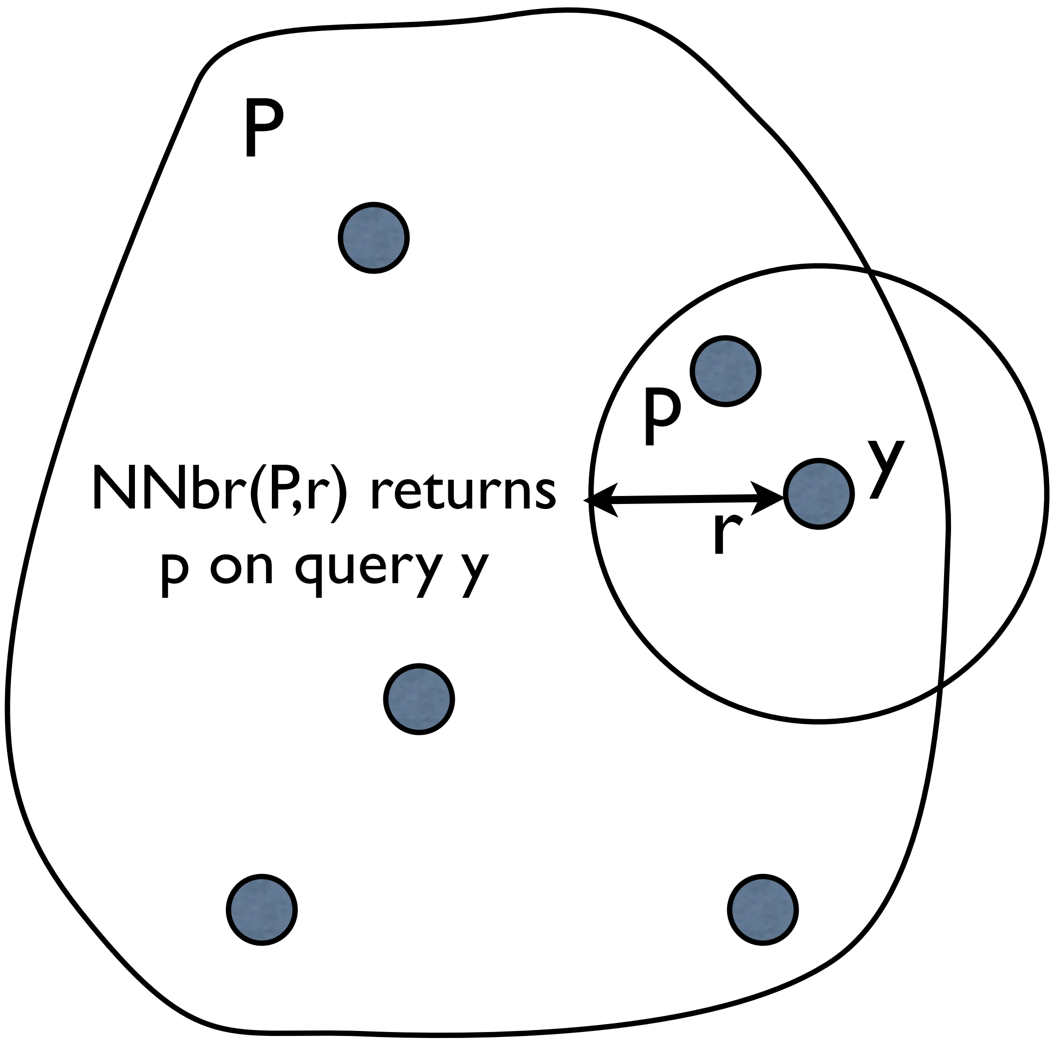


# What We Need to Fix

- This works, but has unbounded complexity.
- We want the number of balls we need to check to be linear.
- We first try limiting the range of the radii of the balls.
- First, we need to figure out how to handle a range of distances.

# Near-Neighbor Data Structure (NNbr)

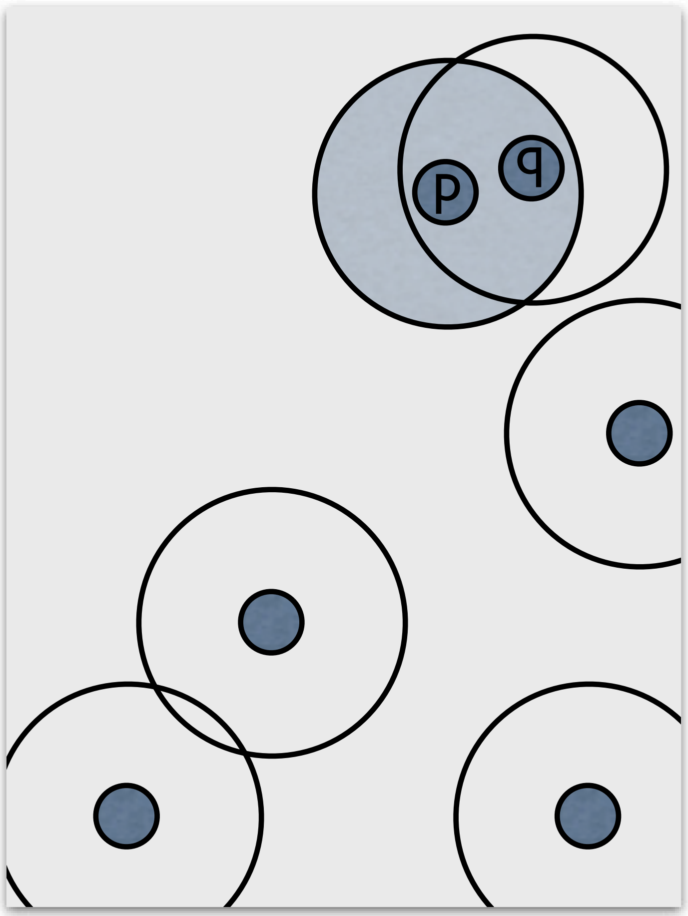
- Let **d**(q,P) be the infinum of **d**(q,p) for $p \in P$.
- NNbr(P,r) is a data structure, such that when given a query point q, it can decide if **d**(q,P) $\leq$ r.
- If **d**(q,P) $\leq$ r, NNbr(P,r) also returns a witness point p such that **d**(q,p) $\leq$ r.



# Near-Neighbor Data Structure (NNbr)

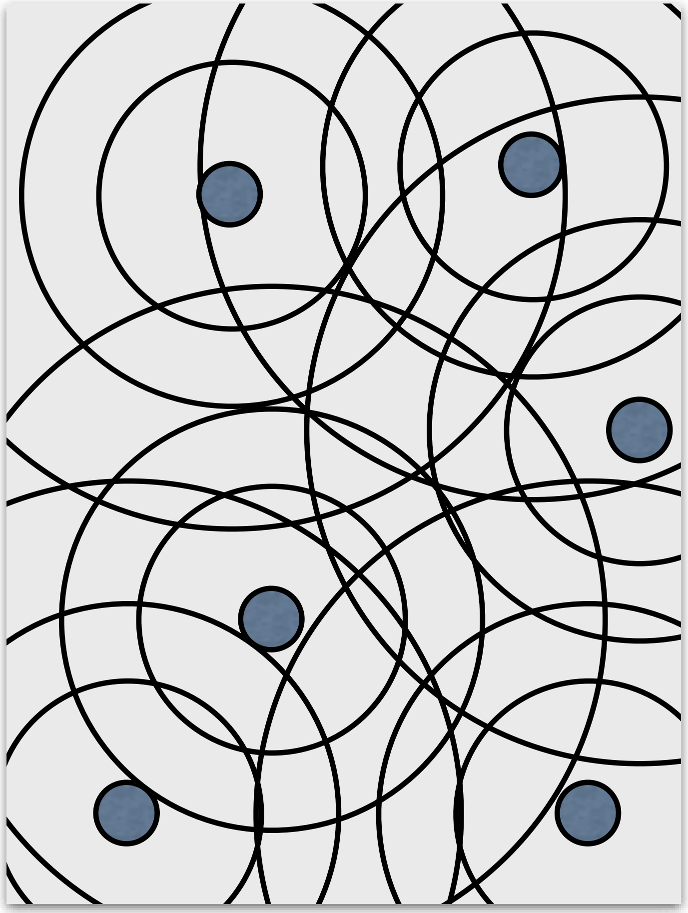
- Can be realized by n balls of radius r around the points of P.

- Perform target ball queries on this set of balls.



# Interval Near-Neighbor Data Structure

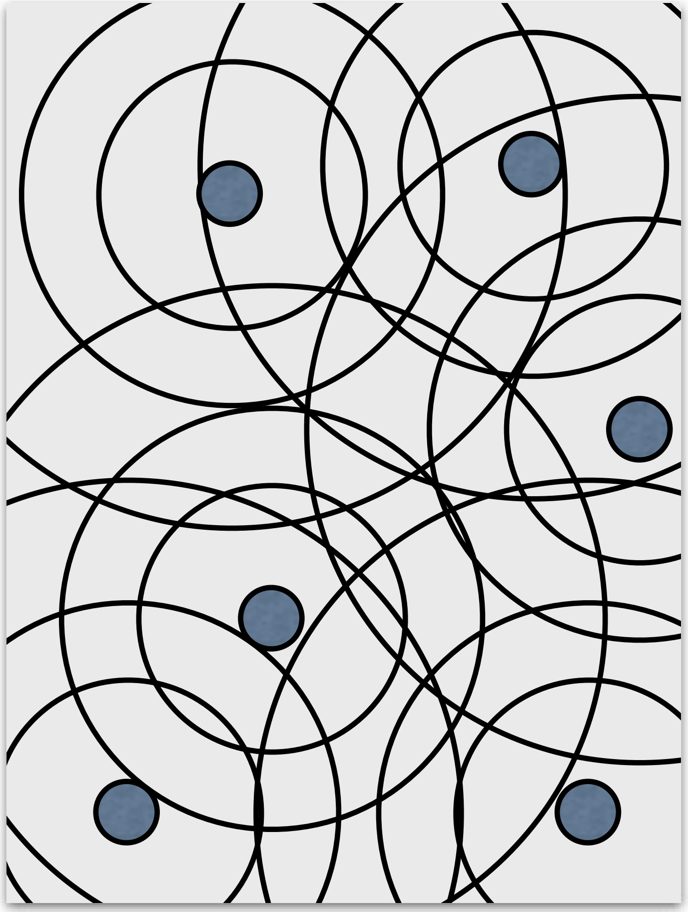
- NNbr data structure with exponential jumps in range.
- $N_i = \text{NNbr}(P, (1+\epsilon)^i a)$
- $M = \log_{1+\epsilon}(b/a)$
- $I(P,a,b,\epsilon) = \{N_0, \ldots, N_M\}$



# Interval Near-Neighbor Data Structure

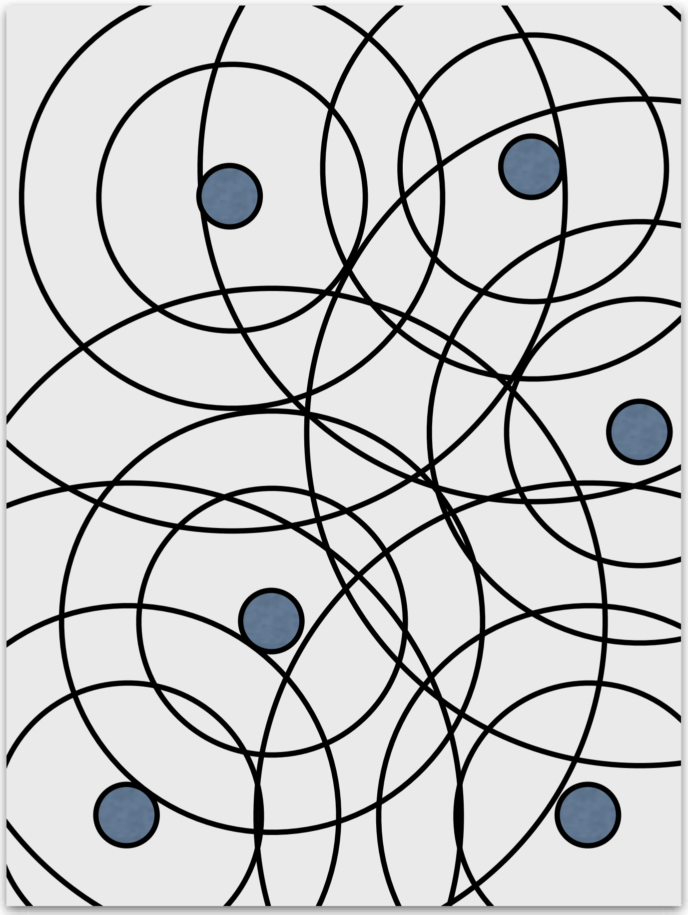
- $\log_{1+\epsilon}(b/a) = O(\log(b/a)/\log(1+\epsilon)) = O(\epsilon^{-1}\log(b/a))$ NNbr data structures.

- $O(\epsilon^{-1} n\log(b/a))$ balls.



# Using Interval NNbr to find ANN

- First check boundaries: O(1) NNbr queries, O(n) target ball queries.
- Then, do binary search on the M NNbr's. This is O(log($\epsilon^{-1}$log(b/a))) NNbr queries, or O(nlog($\epsilon^{-1}$log(b/a))) target ball queries.
- Fast if b/a small.

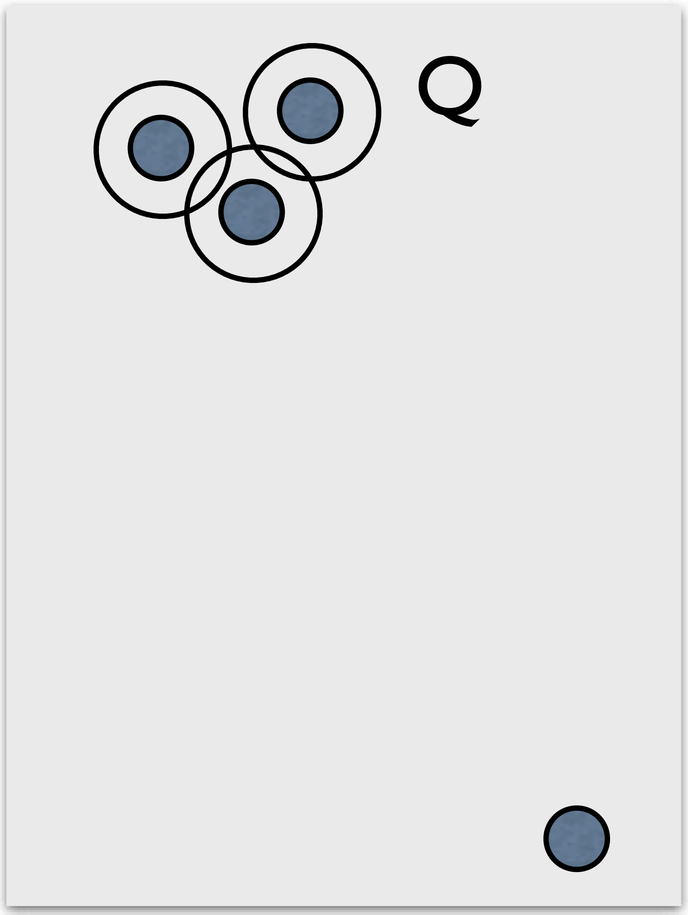


# Faraway Clusters of Points

- Let Q be a set of m points.

- Let U be the union of the balls of radius r around the points of Q

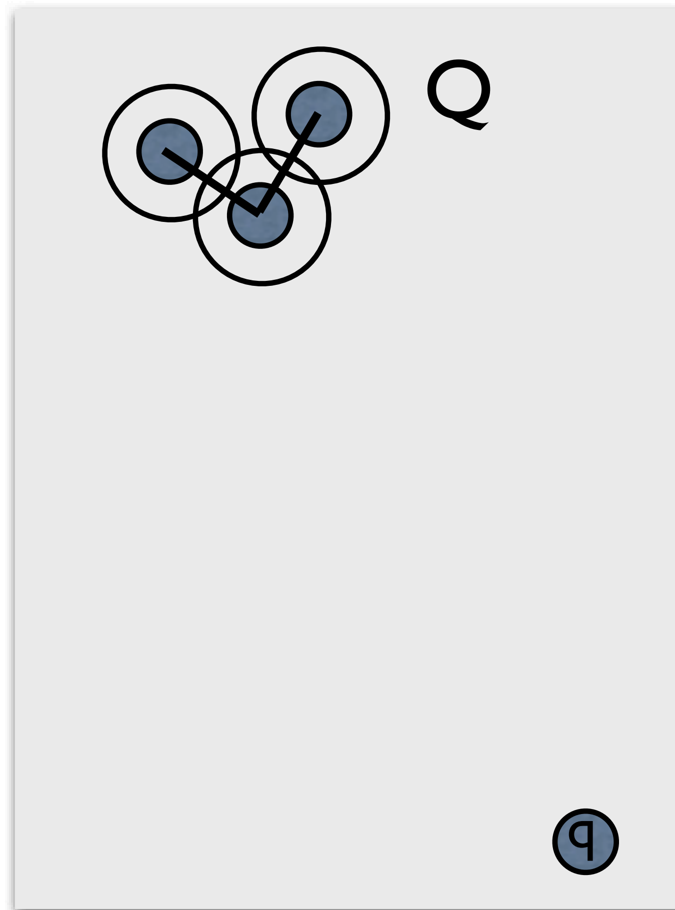
- Suppose U is connected.



# Faraway Clusters of Points

- Any two points p,q in Q are in distance ≤ 2r(m-1) from each other.

- If **d**(q,Q) > 2mr/δ, any point of Q is a (1+δ)-ANN of q in Q.



# Faraway Clusters of Points

- Let s be the closest point in Q to q.
- Let p be any member of Q
- $2mr/\delta < \mathbf{d}(q,s) \leq \mathbf{d}(q,p) \leq \mathbf{d}(q,s) + \mathbf{d}(s,p) \leq \mathbf{d}(q,s) + 2mr \leq (1+\delta)\mathbf{d}(q,s)$