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Abstract: The notion of ε-sample, as introduced by Amenta and Bern, has proven to be a key concept in the theory of sampled surfaces. Of particular interest is the fact that, if E is an ε-sample of a smooth surface S for a sufficiently small ε, then the Delaunay triangulation of E restricted to S is a good approximation of S, both in a topological and in a geometric sense. Hence, if one can construct an ε-sample, one also gets a good approximation of the surface. Moreover, correct reconstruction is ensured by various algorithms.

In this paper, we introduce the notion of loose ε-sample. We show that the set of loose ε-samples contains and is asymptotically identical to the set of ε-samples. The main advantage of loose ε-samples over ε-samples is that they are easier to check and to construct. We also present a simple algorithm that constructs provably good surface samples and meshes.

Key-words: Surface mesh generation, sampling, computational geometry

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Une condition effective pour l'échantillonnage certifié de surfaces

Résumé : La notion d'ε-échantillon introduite par Amenta et Bern est un concept clé de la théorie des surfaces échantillonnées. Un résultat important est en particulier le suivant : si E est un ε-échantillon d'une surface lisse S pour un ε assez petit, alors la triangulation de Delaunay restreinte à S est une bonne approximation de S, à la fois du point de vue topologique et du point de vue géométrique. Ainsi, si on sait construire un ε-échantillon, on obtient du même coup une bonne approximation de la surface. De plus, il existe plusieurs méthodes pour reconstruire une approximation certifiée de S à partir de E.

Dans cet article, on introduit la notion d' ε-échantillon lâche. On montre que cette notion est plus générale que celle d'ε-échantillon et que les deux notions sont identiques asymptotiquement. L'avantage principal des ε-échantillons lâches est d'être plus simples à construire. Nous proposons un algorithme simple et général qui construit des ε-échantillons lâches et des maillages certifiés.

Mots-clés : Maillage de surface, échantillonnage, géométrie algorithmique

1 Introduction

Meshing and reconstructing surfaces are two fundamental problems in geometry processing. In surface reconstruction, a finite set of points E on a surface S is given and one wants to compute a good approximation of S from E. This is of course only possible if E is a good sample of S in some sense. In surface mesh generation, the problem is somehow opposite. A surface S is known and we want to compute a triangulated surface that suitably approximates S. Clearly, the vertices of the triangulated surface have to sample correctly S. Hence, in both applications and also in many others, including the new arena of point set surfaces [1], the notion of good sample is crucial.

The notion of ε-sample, as introduced by Amenta and Bern [2], has proven to be a key concept in the theory of sampled surfaces. Roughly, an ε-sample E of a surface S is a (non necessarily uniform) point set that is sufficiently dense with respect to the *distance to the medial axis* of S – see section 2. Of particular interest is the fact that if E is an ε-sample of a smooth surface S for a sufficiently small ε, the Delaunay triangulation of E restricted to S, $\mathrm{Del}_{|S}(E)$, is a good approximation of S, both in a topological and in a geometric sense (see section 2 for more details). Hence, given an ε-sample of a surface, it is easy to get a good approximation of the surface.

This result (and variants of it) plays a central role in the analysis of all surface reconstruction algorithms that offer theoretical guarantees [8]. In particular, if E is an ε-sample of a smooth surface S for a sufficiently small ε, these algorithms can reconstruct a surface that has the same topology type as S and is close to S.

One drawback of the concept of ε-sample is the fact that it is difficult to check whether a sample is an ε-sample of a given surface, and even more difficult to construct a (preferably sparse) ε-sample of a given surface. This is due to the fact that a direct application of the definition of an ε-sample leads to complicated operations like cutting the surface with balls.

In this paper, we introduce the notion of loose ε-sample. The set of loose ε-samples contains and is asymptotically identical to the set of ε-samples. The main advantage of loose ε-samples over ε-samples is that they are easier to check and to construct. Indeed, checking that a sample is a loose ε-sample reduces to checking whether a finite number of spheres are small enough with respect to the distance from their centers to the medial axis of the surface.

We also present a construction algorithm which is a variant of Chew's surface meshing algorithm [12]. Given a smooth closed surface S, the algorithm generates a sparse ε-sample E and at the same time a triangulated surface $\mathrm{Del}_{|S}(E)$. The triangulated surface has the same topological type as S, is close to S for the Hausdorff distance (see theorem 4.8) and can provide good approximations of normals, areas and curvatures. A remarkable feature of the algorithm is that the surface needs only to be known through an oracle that, given a line segment, detects whether the segment intersects the surface and, in the affirmative, returns an intersection point and the distance to the medial axis at that point (or any smaller non-zero quantity). This makes the algorithm useful in a wide variety of contexts and for a large class of surfaces.

The paper is organized as follows. In section 2, we recall useful concepts and introduce the notion of loose ε-sample. In section 3, we present some local properties of loose ε-samples that are used in section 4 to establish our main results. We prove that, for sufficiently small ε, $\mathrm{Del}_{|S}(E)$ is a 2-manifold without boundary that is ambient isotopic to S and whose Hausdorff distance to S is $O(\varepsilon^2)$. We also prove that S is covered by the so-called surface Delaunay balls, and that loose ε-samples are $\varepsilon(1+16\varepsilon)$-samples. In section 5, we bound the size of loose ε-samples. As an application of our results, we present in section 6 our surface mesh generator.

2 Definitions and preliminary observations

In the paper, S denotes a compact, orientable, twice-differentiable surface without boundary. S will be called a smooth closed surface for short. By $\overrightarrow{n}(p)$ we denote the surface normal at point $p \in S$, and by $T(p)$ the plane tangent to S at p.

Our analysis uses the fact that locally a smooth surface is the graph of a function. More precisely, given an orthonormal frame (O, x, y, z) of $\mathbb{R}^3$, a subset of $\mathbb{R}^3$ is said to be *xy-monotone* if it is the graph of a function of the two variables x and y. A *terrain* is a surface that is xy-monotone in some frame (O, x, y, z)

of $\mathbb{R}^3$. Similarly, given an orthonormal frame (O, x, y) of $\mathbb{R}^2$, a subset of $\mathbb{R}^2$ is said to be *x-monotone* if it is the graph of a function of variable x.

2.1 Restricted Delaunay triangulation

In the paper, E denotes a finite point sample of S and $\mathrm{Del}(E)$ the 3-dimensional Delaunay triangulation of E. By $\mathcal{V}(E)$ we denote the set of the edges of the Voronoi diagram of E.

We call Delaunay triangulation of E *restricted to* S, and we note $\mathrm{Del}_{|S}(E)$, the sub-complex of $\mathrm{Del}(E)$ that consists of the facets of $\mathrm{Del}(E)$ whose dual Voronoi edges intersect S. An edge or vertex of $\mathrm{Del}(E)$ belongs to $\mathrm{Del}_{|S}(E)$ if it is incident to at least one facet of $\mathrm{Del}_{|S}(E)$. Notice that we depart from the usual definition [12, 17] and do not consider vertices and edges with no incident facet of $\mathrm{Del}_{|S}(E)$.

A facet (resp. edge, vertex) of $\mathrm{Del}_{|S}(E)$ is called a *restricted Delaunay facet* (resp. *restricted Delaunay edge*, *restricted Delaunay vertex*). For a restricted Delaunay facet f, we call *surface Delaunay ball* of f any ball circumscribing f centered at some point of $S \cap f^*$, where f^* is the Voronoi edge dual to f. We call *surface Delaunay patch* the intersection of a surface Delaunay ball with S. Notice that the centers of the surface Delaunay balls are precisely the intersection points of S and $\mathcal{V}(E)$.

2.2 ε-samples and loose ε-samples

The *medial axis* of S, denoted by M, is the topological closure of the set of points of $\mathbb{R}^3$ that have more than one nearest neighbour in S.

For a point $x \in \mathbb{R}^3$, we call *distance to the medial axis* at x, and write $d_M(x)$, the Euclidean distance from x to the medial axis of S.

As noticed by Amenta and Bern [2], d_M is 1-Lipschitz, i.e. $|d_M(x) - d_M(y)| \leq \|x - y\|$.

We define $d_M^{\inf} = \inf\{d_M(x),\ x \in S\}$ and $d_M^{\sup} = \sup\{d_M(x),\ x \in S\}$. Since S is a smooth closed surface, both $d_M^{\inf}$ and $d_M^{\sup}$ are finite and strictly positive constants.

We borrow from Amenta and Bern [2] the notion of ε-sample, defined below. In the whole paper, $B(c, r)$ denotes the ball of center c and radius r.

Definition 2.1 *E is an* ε-sample *of S if* $\forall x \in S,\ E \cap B(x, \varepsilon\ d_M(x)) \neq \emptyset$.

For sufficiently small values of ε, ε-samples enjoy many beautiful properties. We recall the most important ones in our context.

- *Normals:* the angle between the normal to a facet f of $\mathrm{Del}_{|S}(E)$ and the normal to S at the vertices of f is $O(\varepsilon)$ [2].
- *Area:* the area of $\mathrm{Del}_{|S}(E)$ approximates the area of S [23].
- *Curvatures:* the curvature tensor of S can be estimated from $\mathrm{Del}_{|S}(E)$ [13].
- *Homeomorphism:* $\mathrm{Del}_{|S}(E)$ is homeomorphic to S [2].
- *Hausdorff distance:* the Hausdorff distance between S and $\mathrm{Del}_{|S}(E)$ is $O(\varepsilon)$ [9]. In this paper, we give an $O(\varepsilon^2)$ bound (theorem 4.8).
- *Reconstruction:* several algorithms can reconstruct from E a surface that is homeomorphic [2, 3, 8, 15] or even ambient isotopic [5] to S.

We will show that these properties hold for loose ε-samples as well.

Definition 2.2 *E is a* loose ε-sample *of S if* $\forall x \in S \cap \mathcal{V}(E),\ E \cap B(x, \varepsilon\ d_M(x)) \neq \emptyset$.

Since the centers of the surface Delaunay balls are precisely the intersection points of S with the Voronoi edges, E is a loose ε-sample if and only if every surface Delaunay ball $B(c, r)$ has a radius of at most $\varepsilon\ d_M(c)$.

ε-samples and loose ε-samples are closely related but not identical concepts. The next lemma follows from definitions 2.1 and 2.2.

Lemma 2.3 *If E is an ε-sample, then it is a loose ε-sample.*

The converse is true asymptotically, as we will see in section 4.3 (corollary 4.13).

2.3 Other notations

The following constants are used in the paper:

- ε_0 is the only positive root of equation $\frac{2\varepsilon}{1-8\varepsilon} + \arcsin \frac{\varepsilon}{1-\varepsilon} - \frac{\pi}{4} = 0$. $\varepsilon_0 \approx 0.091$.
- ε_1 is the only positive root of equation $\frac{2\varepsilon}{1-7\varepsilon} + \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} - \frac{\pi}{4} = 0$. $\varepsilon_1 \approx 0.096$.
- $\varepsilon_2 = \frac{\pi}{4+9\pi} \approx 0.097$.
- ε_3 is the only positive root of equation $\frac{\varepsilon}{1-5\varepsilon} + \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} - \frac{\pi}{2} = 0$. $\varepsilon_3 \approx 0.17$.
- ε_4 is the only positive root of equation $\frac{2\varepsilon}{1-4\varepsilon} + \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} - \frac{\pi}{4} = 0$. $\varepsilon_4 \approx 0.12$.

We also use the notation $(\overrightarrow{u}, \overrightarrow{v})$ to denote the modulus of the angle (measured in $[-\pi, \pi]$) between vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ of $\mathbb{R}^3$, and $\overrightarrow{u}.\overrightarrow{v}$ to denote their dot-product.

3 Local properties of loose ε-samples

In this section, we prove that surface Delaunay balls of sufficiently small radii keep important properties of planar disks. In particular, we show that they intersect S along topological disks whose boundaries pairwise intersect in at most two points (proposition 3.9).

3.1 Technical lemmas

In this paragraph, we recall several lemmas by Amenta and Bern [2] that will be useful in the remainder of the paper.

Lemma 3.1 *Let f be a facet of $Del_{|S}(E)$. Assume that every surface Delaunay ball $B(c, r)$ of f is such that $r \leq \rho\ d_M(c)$, with $\rho < \frac{1}{7}$. Let a be a vertex of f. If a has an inner angle of at least $\pi/3$, then the smaller angle between the line normal to f and the normal to S at a is at most $\arcsin \frac{\rho\sqrt{3}}{1-\rho}$. Otherwise, the smaller angle between the line normal to f and the normal to S at a is at most $\frac{2\rho}{1-7\rho} + \arcsin \frac{\rho\sqrt{3}}{1-\rho}$.*

Proof The radius of any surface Delaunay ball of f is at most $\frac{\rho}{1-\rho}\ d_M(a)$, thus the proof of lemma 7 of Amenta and Bern [2] holds here. □

Lemma 3.2 *For any two points p and q on S with $\|p - q\| \leq \rho\ d_M(p)$, the smaller angle between the line segment pq and the surface normal at p is at least $\frac{\pi}{2} - \arcsin \frac{\rho}{2}$.*

Lemma 3.3 *For any two points p and q on S with $\|p - q\| \leq \rho\ \min\{d_M(p), d_M(q)\}$, for any $\rho < \frac{1}{3}$, the angle between the normals to S at p and at q is at most $\frac{\rho}{1-3\rho}$.*

We will need the two following corollaries of the above lemmas.

Lemma 3.4 *Let $B = B(c, r)$ and $B' = B(c', r')$ be two distinct balls centered on S, whose bounding spheres ∂B and $\partial B'$ intersect. Assume that their radii are at most $\varepsilon\ d_M(c)$ and $\varepsilon\ d_M(c')$ respectively, with $\varepsilon < \frac{1}{8}$. Then there exists a vector $\overrightarrow{v}$ orthogonal to $\overrightarrow{cc'}$, such that the angle between $\overrightarrow{v}$ and the normal to S at any point of $S \cap B(c, 2r)$ is at most $\frac{2\varepsilon}{1-8\varepsilon} + \arcsin \frac{\varepsilon}{1-\varepsilon}$. Hence, if $\varepsilon \leq \varepsilon_0$, this angle is at most $\frac{\pi}{4}$.*

Proof Define $B^+ = B(c, 2r)$. We have

$$\forall x \in B^+ \cap S,\ \|x - c\| \leq 2r \leq 2\varepsilon\ d_M(c) \tag{1}$$

thus

$$\forall x \in B^+ \cap S,\ d_M(x) \begin{array}{l} \geq d_M(c) - \|x - c\| \\ \geq (1 - 2\varepsilon)\ d_M(c) \end{array} \tag{2}$$

(1) and (2) give

$$\forall x \in B^+ \cap S,\ \|x - c\| \leq \frac{2\varepsilon}{1-2\varepsilon}\ \min\{d_M(c), d_M(x)\}$$

which implies, according to lemma 3.3,

$$\forall x \in B^+ \cap S,\ (\overrightarrow{n}(x), \overrightarrow{n}(c)) \leq \frac{2\varepsilon/(1-2\varepsilon)}{1-6\varepsilon/(1-2\varepsilon)} = \frac{2\varepsilon}{1-8\varepsilon}$$

We have $\|c - c'\| \leq r + r' \leq \varepsilon\ d_M(c) + \varepsilon\ d_M(c') \leq 2\varepsilon\ d_M(c) + \varepsilon\ \|c - c'\|$, *i.e.* $\|c - c'\| \leq \frac{2\varepsilon}{1-\varepsilon}\ d_M(c)$. Thus, lemma 3.2 tells that

$$\min\{(\overrightarrow{cc'}, \overrightarrow{n}(c)), (\overrightarrow{n}(c), \overrightarrow{cc'})\} \geq \frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon} \tag{3}$$

Inside plane $(c, \overrightarrow{cc'}, \overrightarrow{n}(c))$, let $\overrightarrow{v}$ be the unitary vector that is orthogonal to $\overrightarrow{cc'}$ and has a positive scalar product with $\overrightarrow{n}(c)$. According to (3) we have $(\overrightarrow{n}(c), \overrightarrow{v}) \leq \arcsin\frac{\varepsilon}{1-\varepsilon}$. Thus,

$$\begin{aligned} \forall x \in S \cap B^+,\ (\overrightarrow{n}(x), \overrightarrow{v}) &\leq (\overrightarrow{n}(x), \overrightarrow{n}(c)) + (\overrightarrow{n}(c), \overrightarrow{v}) \\ &\leq \tfrac{2\varepsilon}{1-8\varepsilon} + \arcsin\tfrac{\varepsilon}{1-\varepsilon} \end{aligned}$$

□

Lemma 3.5 *Let $B = B(c, r)$ be a ball centered at $c \in S$ of radius $r \leq \varepsilon\ d_M(c)$, with $\varepsilon < \frac{1}{9}$, and let v be a point of $S \cap B$. The normals to S at v and at any point of $S \cap B(v, 2r)$ make an angle of at most $\frac{2\varepsilon}{1-9\varepsilon}$. Hence, if $\varepsilon < \varepsilon_2$, this angle is less than $\frac{\pi}{2}$ and, by lemma 9.4, $S \cap B(v, 2r)$ is a terrain.*

Proof We have $r \leq \varepsilon\ d_M(c) \leq \varepsilon\ (d_M(v) + \|v - c\|)$, which is at most $\varepsilon\ (d_M(v) + r)$ since v lies in B. Thus, $r \leq \frac{\varepsilon}{1-\varepsilon}\ d_M(v)$. Therefore, $\forall x \in S \cap B(v, 2r)$,

$$\begin{aligned} \|x - v\| &\leq 2r \leq \tfrac{2\varepsilon}{1-\varepsilon}\ d_M(v) \\ &\leq \tfrac{2\varepsilon}{1-\varepsilon}\,(d_M(x) + \|x - v\|) \end{aligned}$$

which implies $\|x - v\| \leq \frac{2\varepsilon}{1-3\varepsilon}\ d_M(x)$. It follows that $\|x - v\| \leq \rho \min\{d_M(x), d_M(v)\}$, with $\rho = \frac{2\varepsilon}{1-3\varepsilon}$. Thus, according to lemma 3.3,

$$(\overrightarrow{n}(x), \overrightarrow{n}(v)) \leq \frac{\rho}{1-3\rho} = \frac{2\varepsilon}{1-9\varepsilon}$$

□

3.2 Topological disks and terrains

Lemma 3.6 ([7]) *Let B be a ball that intersects S. If the intersection is not a topological disk, then B contains a point of the medial axis of S. As a consequence, if E is a loose ε-sample, with $\varepsilon < 1$, then surface Delaunay patches are topological disks.*

Lemma 3.7 *If E is a loose ε-sample, with $\varepsilon < \varepsilon_2$, then, for every surface Delaunay ball $B = B(c, r)$, for any point $x \in S \cap B$, $S \cap B(x, 2r)$ is a topological disk and a terrain.*

Proof Since E is a loose ε-sample, we have $r \leq \varepsilon\ d_M(c) \leq \varepsilon(d_M(x) + \|x - c\|) \leq \varepsilon(d_M(x) + r)$, that is, $r \leq \frac{\varepsilon}{1-\varepsilon}\ d_M(x)$. Thus, $2r < d_M(x)$ since $\varepsilon < \varepsilon_2 < \frac{1}{3}$. According to lemma 3.6, $S \cap B(x, 2r)$ is thus a topological disk. The fact that $S \cap B(x, 2r)$ is a terrain follows from lemma 3.5, since $\varepsilon < \varepsilon_2$. □

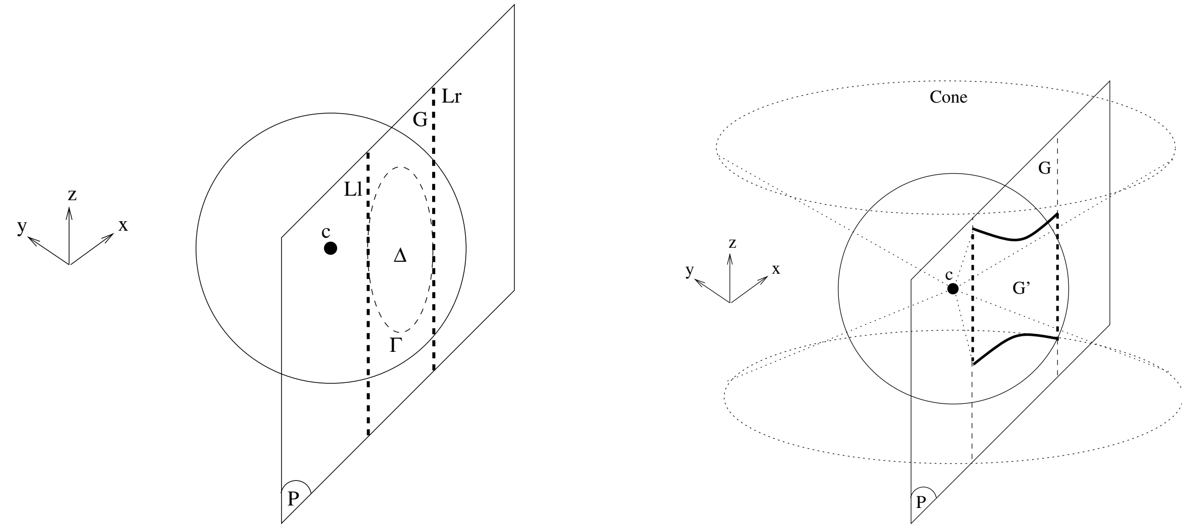


Figure 1: Definition of G and G'

3.3 Pseudo-disks

Definition 3.8 *Topological disks are pseudo-disks if they pairwise intersect along topological disks (that may be empty or reduced to a point) and if their boundaries pairwise intersect in at most two points.*

Observe that the boundaries of two pseudo-disks either do not intersect, or intersect in one point tangentially, or intersect in two points transversally.

Proposition 3.9 *If E is a loose ε-sample, with $\varepsilon \leq \varepsilon_0$, then surface Delaunay patches are pseudo-disks.*

Proof Let $B = B(c, r)$ and $B' = B(c', r')$ be two surface Delaunay balls. According to lemma 3.6, $D = B \cap S$ and $D' = B' \cap S$ are topological disks, since $\varepsilon \leq \varepsilon_0 < 1$. Their boundaries C and C' are topological circles. Let us assume that balls B and B' intersect, the other case being trivial. Notice that none of them can be contained in the other one, since they are Delaunay balls. Thus, their bounding spheres ∂B and $\partial B'$ also intersect. Let Γ be the circle $\partial B \cap \partial B'$, ρ its radius ($\rho < \min\{r, r'\}$) and P its supporting plane. We define $\Delta = B \cap P$ and notice that $\Gamma = \partial\Delta$. Since S is a closed surface, we have $C \subset \partial B$ and $C' \subset \partial B'$, which implies that

$$C \cap C' \subseteq S \cap \Gamma \tag{4}$$

Let $B^+ = B(c, 2r)$. Since $\varepsilon \leq \varepsilon_0$, by lemma 3.4 there exists a vector $\overrightarrow{v}$ orthogonal to $\overrightarrow{cc'}$ such that

$$\forall x \in S \cap B^+,\ (\overrightarrow{n}(x), \overrightarrow{v}) \leq \frac{\pi}{4} \tag{5}$$

Let us choose in $\mathbb{R}^3$ a reference frame of origin c, of y-axis directed along $\overrightarrow{c'c}$, and of z-axis directed along $\overrightarrow{v}$. We call L_l and L_r the two lines of P, parallel to the z-axis, that are tangent to Γ. The region of P bounded by L_l and L_r is called G (see figure 1). In the following, ξ denotes $S \cap B^+ \cap G$.

Lemma 3.10 *ξ is a connected x-monotone arc.*

Proof According to (5), we have $\forall x \in S \cap B^+$, $(\overrightarrow{n}(x), \overrightarrow{v}) \leq \frac{\pi}{4}$. Thus, by lemma 9.4, $B^+ \cap S$ is xy-monotone, which implies that ξ is x-monotone. Moreover, according to lemma 9.5, $B^+ \cap S$ lies outside the cone of apex $c \in S$, of vertical axis and of half-angle $\frac{\pi}{4}$. The equation of the cone in our frame is $z^2 = x^2 + y^2$. It intersects P along two hyperbolic arcs of equations $z = \pm\sqrt{x^2 + d^2}$, where $d \leq r$ is the distance from c to P. Consider the subregion G' of G that is bounded vertically by the two hyperbolic arcs (see figure 1). Since

$S \cap B^+$ lies outside the cone, ξ is included in G'. The points of G' that are farthest from c are the points $(\pm\rho, -d, \pm\sqrt{\rho^2 + d^2})$. Their distance to c is

$$\sqrt{2(\rho^2 + d^2)} < 2r$$

In other words, $G' \subset \mathrm{int}(B^+)$. It follows that ξ is included in $\mathrm{int}(B^+)$ and cannot intersect ∂B^+. Its endpoints must then lie on the vertical lines L_r and L_l. But there can be only one endpoint per vertical line, since ξ is x-monotone. Hence, ξ has at most two endpoints and is thus connected. □

Lemma 3.11 $|S \cap \Gamma| \leq 2$.

Proof Let us assume for a contradiction that $|S \cap \Gamma| > 2$. First, we show that there exists a point where the curvature of ξ is high and hence the distance to the medial axis M is small. Then we work out a contradiction with the fact that E is a loose ε-sample, with $\varepsilon \leq \varepsilon_0$.

Claim 1 *There exists a point q at which the curvature of ξ is at least $\frac{1}{\rho}$.*

Proof We made the assumption that $|S \cap \Gamma| > 2$. Since $\Gamma \subset G$ and $\Gamma \subset B^+$, ξ also intersects Γ more than twice. And since ξ is connected by lemma 3.10, there is a subarc ab of ξ that lies outside Δ and whose endpoints a and b lie on Γ. This subarc may be reduced to a point ($a = b$), since ξ may be tangent to Γ. But in this case, in the vicinity of a, ξ is locally included in Δ and tangent to Γ at a. Thus, its curvature at a is at least $\frac{1}{\rho}$, which proves the claim with $q = a$. So now we assume that arc ab of ξ is not reduced to a point. Since ξ is x-monotone by lemma 3.10, a and b lie on the same half of Γ, upper half or lower half (say upper half). Thus, the smaller arc of Γ that joins a and b is also x-monotone. Then, by lemma 9.3, there is a point q of arc ab of ξ at which the curvature of ξ is at least $\frac{1}{\rho}$, which proves the claim. □

Claim 2 $d_M(q) \leq \rho\sqrt{2}$.

Proof Let $\vec{n}_\xi(q)$ be the normal to planar curve ξ at point q. By inequation (5), $\vec{n}(q)$ is not orthogonal to P, thus $\vec{n}_\xi(q)$ is oriented along the projection of $\vec{n}(q)$ onto P. Hence, by lemma 9.2, we have $(\vec{n}(q), \vec{n}_\xi(q)) \leq (\vec{n}(q), \vec{v})$ which is at most $\frac{\pi}{4}$ by inequation (5). According to theorem 9.1, we then have at q

$$\mathit{II}(\xi', \xi') \geq \cos\frac{\pi}{4}\ \|\xi''\|$$

ξ' is the unit tangent vector of ξ at q and $\|\xi''\|$ is the curvature of ξ at q, which is more than $\frac{1}{\rho}$ according to claim 1. So, at q we have

$$\mathit{II}(\xi', \xi') \geq \frac{1}{\rho\sqrt{2}} \tag{6}$$

Recall that II is a symmetric bilinear form, thus it can be diagonalized in an orthonormal frame, and its eigenvalues are the minimum and maximum curvatures of S at q. Let us call these values $\kappa_{min}(q)$ and $\kappa_{max}(q)$ respectively. Since ξ' is a unit vector, we have $\mathit{II}(\xi', \xi') \leq \max\{|\kappa_{min}(q)|, |\kappa_{max}(q)|\}$. It follows, according to (6), that $\max\{|\kappa_{min}(q)|, |\kappa_{max}(q)|\} \geq \frac{1}{\rho\sqrt{2}}$, or, equivalently, that the minimal radius of curvature of S at q is at most $\rho\sqrt{2}$. The result follows. □

The end of the proof of the lemma is immediate. We have

$$\begin{aligned} d_M(c) &\leq d_M(q) + \|c - q\| \\ &\leq \rho\sqrt{2} + 2r \\ &\leq r(\sqrt{2} + 2) \end{aligned}$$

So, the radius of ball B is at least $\frac{1}{\sqrt{2}+2}\ d_M(c)$, which contradicts the assumption that E is a loose ε-sample, with $\varepsilon \leq \varepsilon_0 < \frac{1}{\sqrt{2}+2}$. □

From lemma 3.11, it immediately follows that $|C \cap C'| \leq 2$, by (4).

Lemma 3.12 *$S \cap \Delta$ is not reduced to two points.*

Proof Let us assume that S intersects Δ in two points exactly, say a and b. Then, the subarc of ξ that joins points a and b lies outside Δ. It follows, by the same reasoning as in the proof of claim 1, that there exists some point q of ξ at which the curvature of ξ is at least $\frac{1}{\rho}$. It follows by claim 2 that $d_M(q) \leq \rho\sqrt{2}$, which leads to a contradiction, as in the end of the proof of lemma 3.11. □

It follows from the above lemmas that D and D' intersect along a topological disk. The result is clear if $D \subseteq D'$ or if $D' \subseteq D$. Otherwise, we have $|C \cap C'| \leq 2$, by lemma 3.11. If $|C \cap C'| = 0$, then $D \cap D'$ is empty. If $|C \cap C'| = 1$, then $D \cap D'$ is reduced to a point. If $|C \cap C'| = 2$, then $D \cap D'$ is either a topological disk or equal to $C \cap C'$. But if $D \cap D' = C \cap C'$, then $S \cap \Delta = C \cap C'$ since $C \cap C' \subseteq S \cap \Delta \subseteq D \cap D'$. This contradicts lemma 3.12. Hence, $D \cap D'$ is not equal to $C \cap C'$ and is therefore a topological disk. This ends the proof of proposition 3.9. □

4 Global properties of loose ε-samples

In this section, E is a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0$. We prove that $\mathrm{Del}_{|S}(E)$ is a manifold without boundary (theorem 4.5), ambient isotopic to S (corollary 4.7), at Hausdorff distance $O(\varepsilon^2)$ from S (theorem 4.8). From the latter we deduce that E is an $\varepsilon(1+16\varepsilon)$-sample of S (corollary 4.13). We also prove that the surface Delaunay balls cover S (theorem 4.15).

4.1 Manifold

We first prove that every edge of $\mathrm{Del}_{|S}(E)$ is incident to exactly two facets of $\mathrm{Del}_{|S}(E)$. We then prove that every vertex of $\mathrm{Del}_{|S}(E)$ has only one *umbrella*. An umbrella of a vertex v is a subset of facets of $\mathrm{Del}_{|S}(E)$ incident to v whose adjacency graph is a cycle.

Lemma 4.1 *The dual of a facet of $\mathrm{Del}_{|S}(E)$ intersects S only once, and transversally.*

Proof Let f be a facet of $\mathrm{Del}_{|S}(E)$, and f^* its dual Voronoi edge. We denote by a the vertex of f that has the largest inner angle. We have $\hat{a} \geq \frac{\pi}{3}$, and since $\varepsilon \leq \varepsilon_0 < \frac{1}{7}$, lemma 3.1 says that

$$(\overrightarrow{n}(a), \overrightarrow{n}_f) \leq \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} \tag{7}$$

where $\overrightarrow{n}_f$ denotes the unitary vector orthogonal to f that makes the smaller angle with $\overrightarrow{n}(a)$. Let B_a be the ball $B(a, \frac{\varepsilon}{1-\varepsilon}\ d_M(a))$. For any surface Delaunay ball $B(c,r)$ that circumscribes f, we have

$$\begin{aligned} \|c-a\| = r \quad &\leq \varepsilon\ d_M(c) \\ &\leq \varepsilon\ \ (d_M(a) + \|c-a\|) \end{aligned}$$

Hence, $\|c-a\| \leq \frac{\varepsilon}{1-\varepsilon}\ d_M(a)$. In other words, every center of surface Delaunay ball of f lies in B_a. In addition, we have

$$\begin{aligned} \forall x \in B_a \cap S,\ \ \|x-a\| \quad &\leq \tfrac{\varepsilon}{1-\varepsilon}\ d_M(a) \\ &\leq \tfrac{\varepsilon}{1-\varepsilon}\ (d_M(x) + \|x-a\|) \end{aligned}$$

which implies $\|x-a\| \leq \frac{\varepsilon}{1-2\varepsilon}\ d_M(x)$. According to lemma 3.3, we then have $\forall x \in B_a \cap S$,

$$(\overrightarrow{n}(x), \overrightarrow{n}(a)) \leq \frac{\varepsilon/(1-2\varepsilon)}{1-3\varepsilon/(1-2\varepsilon)} = \frac{\varepsilon}{1-5\varepsilon} \tag{8}$$

(7) and (8) give

$$\begin{aligned} \forall x \in B_a \cap S,\ \ (\overrightarrow{n}(x), \overrightarrow{n}_f) \quad &\leq (\overrightarrow{n}(x), \overrightarrow{n}(a)) + (\overrightarrow{n}(a), \overrightarrow{n}_f) \\ &\leq \tfrac{\varepsilon}{1-5\varepsilon} + \arcsin \tfrac{\varepsilon\sqrt{3}}{1-\varepsilon} \end{aligned}$$

which is less than $\frac{\pi}{2}$ since $\varepsilon \leq \varepsilon_0 < \varepsilon_3$. Thus, by lemma 9.4, $B_a \cap S$ is a terrain over the plane Π_f that supports f. Since f^* is orthogonal to Π_f, it cannot intersect $B_a \cap S$ more than once, nor tangentially. And since every center of surface Delaunay ball of f lies in B_a, f^* cannot intersect S more than once, nor tangentially. $\square$

For a restricted Delaunay facet f, we denote by $B_f = (c_f, r_f)$ the corresponding surface Delaunay ball. The surface Delaunay patch of f, $S \cap B_f$, is denoted by D_f. We define $C_f = \partial D_f$.

Lemma 4.2 *Let f and f' be restricted Delaunay facets that share an edge. They make a dihedral angle greater than $\frac{\pi}{2}$.*

Proof This result is a consequence of theorem 1 of [22]. The latter states that the dihedral angle is at least $\pi - 2\left(\frac{2\varepsilon}{1-4\varepsilon} + \arcsin\frac{\varepsilon\sqrt{3}}{1-\varepsilon}\right)$, which is greater than $\frac{\pi}{2}$ since $\varepsilon \leq \varepsilon_0 < \varepsilon_4$. $\square$

From lemmas 4.1 and 4.2 we deduce the following result.

Proposition 4.3 *Every edge of $Del_{|S}(E)$ is incident to exactly two facets of $Del_{|S}(E)$.*

Proof Let e be an edge of $\mathrm{Del}_{|S}(E)$. We denote by e^* the Voronoi face dual to e. Since S has no boundary, $S \cap \mathrm{aff}(e^*)$ is a union of simple closed curves, none of which intersects the boundary ∂e^* of e^* tangentially, by lemma 4.1. Thus, by the Jordan curve theorem, each curve of $S \cap \mathrm{aff}(e^*)$ intersects ∂e^* at an even number of points. It follows that S intersects ∂e^* at an even number of points. Moreover, by lemma 4.1, each edge of ∂e^* can be intersected at most once by S. Thus, S intersects an even number of edges of ∂e^*, and e is incident to an even number of restricted Delaunay facets.

In addition, two restricted Delaunay facets incident to e make a dihedral angle greater than $\frac{\pi}{2}$, by lemma 4.2. It follows that e may be incident to at most three restricted Delaunay facets. In conclusion, the number of restricted Delaunay facets incident to e is even, strictly positive (since $e \in \mathrm{Del}_{|S}(E)$) and at most three, hence it is equal to two. $\square$

It follows from the above proposition that the restricted Delaunay facets incident to a vertex of $\mathrm{Del}_{|S}(E)$ form a set of umbrellas.

Proposition 4.4 *Every vertex of $Del_{|S}(E)$ has exactly one umbrella.*

Proof Let a be a vertex of $\mathrm{Del}_{|S}(E)$. Let $F(a)$ be the set of all facets of $\mathrm{Del}_{|S}(E)$ that are incident to a. Let f_a be the facet of $F(a)$ that has the surface Delaunay ball of largest radius. We call r_{f_a} this radius. Then $B(a, 2r_{f_a})$ contains the surface Delaunay balls of all facets of $F(a)$. Moreover, by lemma 3.7, $S \cap B(a, 2r_{f_a})$ is a topological disk and a terrain over some plane Π. We project $S \cap B(a, 2r_{f_a})$ onto Π. This projection preserves topological properties such as pseudo-disks. For simplicity of notations, we shall identify objects with their projection onto Π. Let $F_1(a)$ be an umbrella of a. We call $U_1(a)$ the union of the facets of $F_1(a)$, and $R_1(a)$ the union of the surface Delaunay patches associated with the facets of $F_1(a)$.

Claim $a \in \mathrm{int}(R_1(a))$.

Proof If $a \in \mathrm{int}(U_1(a))$, then it is clear that $a \in \mathrm{int}(R_1(a))$, since surface Delaunay patches are pseudo-disks, by proposition 3.9. Now, let us assume that a lies on the boundary of $U_1(a)$. Let $[av]$ be an edge of the boundary that is incident to a. By proposition 4.3, $[av]$ is incident to two facets of $\mathrm{Del}_{|S}(E)$, say (a, v, x) and (a, v, x'). These facets belong to $F_1(a)$ since they are incident to a, and they both lie on the same side of $[av]$ which is a boundary edge of $U_1(a)$. Thus, since surface Delaunay patches are pseudo-disks by proposition 3.9, either x is included in the interior of the surface Delaunay patch of (a, v, x') or x' is included in the interior of the surface Delaunay patch of (a, v, x), which violates the Delaunay property. $\square$

We now assume for a contradiction that there exists a restricted Delaunay facet $f = (a, b, d) \notin F_1(a)$ that is incident to a. Vertices b and d lie outside $\mathrm{int}(R_1(a))$, whereas a lies in $\mathrm{int}(R_1(a))$, by the above claim. It follows that C_f intersects the boundary of $R_1(a)$, at some point z that lies on the boundary of the surface

Delaunay patch of some facet $f' = (a, b', d')$ of $F_1(a)$. By proposition 3.9, C_f and $C_{f'}$ intersect at points a and z only. By the same proposition, open arcs (a, b') and (a, d') of $C_{f'}$ are included in $\mathrm{int}(R_1(a))$. Since $a \in \mathrm{int}(R_1(a))$, z lies on arc (b', d') of $C_{f'}$. If $z \neq b'$ and $z \neq d'$, then b' and d' lie on different sides of C_f, hence one of them lies in $\mathrm{int}(D_f)$, which violates the Delaunay property. Otherwise (say $z = b'$), d' must lie outside D_f. In this case, consider the facet $f'' = (a, b', d'')$ of $F_1(a)$ that is incident to f' through edge $[a, b']$. By proposition 3.9, C_f intersects arc (b', d'') of $C_{f''}$ at point b' only, thus d' and d'' lie on different sides of C_f. Hence, either d' or d'' lies in $\mathrm{int}(D_f)$, which violates the Delaunay property. □

The next theorem follows from propositions 4.3 and 4.4.

Theorem 4.5 *Let S be a smooth closed surface and E a loose ε-sample of S. If $\varepsilon \leq \varepsilon_0 \approx 0.091$, then $Del_{|S}(E)$ is a 2-manifold without boundary.*

Since $\mathrm{Del}_{|S}(E)$ is a closed 2-manifold embedded in $\mathbb{R}^3$, we can orient the normals of its facets consistently. For instance, they can be chosen so as to point to the unbounded component of $\mathbb{R}^3 \setminus \mathrm{Del}_{|S}(E)$.

4.2 Homeomorphism and ambient isotopy

Let $\pi : \mathbb{R}^3 \to S$ map each point of $\mathbb{R}^3$ to the closest point of S. In [3], the authors have shown that the restriction of π to a 2-simplicial complex W whose vertices lie on S is a homeomorphism between W and S, provided that:

H0 W is a manifold without boundary.
H1 W has a vertex on each connected component of S.
H2 The angle between the oriented normals of any two incident facets of W is less than $\frac{\pi}{2}$.
H3 (Small Triangle Condition) every facet f of W has a circumcircle of radius at most $\frac{1.3\varepsilon}{1-\varepsilon} d_M(a)$, where a is any vertex of f.
H4 (Flat Triangle Condition) the normal to every facet f of W makes an angle of at most $\arcsin \frac{1.3\varepsilon}{1-\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}} \sin\left(2 \arcsin \frac{1.3\varepsilon}{1-\varepsilon}\right)\right)$ with $\overrightarrow{n}(a)$, where a is the vertex with the largest interior angle in f.

We will show that H0, H2, H3 and H4 are satisfied by $W = \mathrm{Del}_{|S}(E)$ in our context. H0 has already been stated for $\mathrm{Del}_{|S}(E)$ in theorem 4.5.

Proof of H2
Let a be a vertex of $\mathrm{Del}_{|S}(E)$ and let $F(a)$ be the umbrella of a. By lemma 3.1, the smaller angle between $\overrightarrow{n}(a)$ and the line normal to any facet of $F(a)$ is at most $\frac{2\varepsilon}{1-7\varepsilon} + \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon}$, which is less than $\frac{\pi}{4}$ since $\varepsilon \leq \varepsilon_0 < \varepsilon_1$. It follows that the angle between $\overrightarrow{n}(a)$ and the oriented normal of the facet is less than $\frac{\pi}{4}$ or greater than $\frac{3\pi}{4}$. Moreover, any two consecutive facets in the umbrella of a make a dihedral angle greater than $\frac{\pi}{2}$, by lemma 4.2, thus the angles between $\overrightarrow{n}(a)$ and the oriented normals of the facets of $F(a)$ are all less than $\frac{\pi}{4}$, or they are all greater than $\frac{3\pi}{4}$. It follows that the angle between the oriented normals of any two facets of $F(a)$ is less than $\frac{\pi}{2}$. □

Proof of H3
Since E is a loose ε-sample, every facet f of $\mathrm{Del}_{|S}(E)$ has a surface Delaunay ball $B_f = B(c_f, r_f)$ of radius $r_f \leq \varepsilon\, d_M(c_f)$. Let a be any vertex of f. We have $d_M(c_f) \leq d_M(a) + \|a - c_f\| \leq d_M(a) + r_f$, thus $r_f \leq \frac{\varepsilon}{1-\varepsilon}\, d_M(a)$. It follows that the circumcircle of f has a radius of at most $\frac{\varepsilon}{1-\varepsilon}\, d_M(a) \leq \frac{1.3\varepsilon}{1-\varepsilon}\, d_M(a)$. Since $\varepsilon \leq \varepsilon_0$, the radius is at most $\frac{\varepsilon_0}{1-\varepsilon_0}\, d_M(a) \approx 0.1\, d_M(a)$. □

Proof of H4
Let $f \in \mathrm{Del}_{|S}(E)$ and a be the vertex of f with the largest inner angle. By lemma 3.1, we have $(\overrightarrow{n}_f, \overrightarrow{n}(a)) \leq \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon}$, which is at most $\arcsin \frac{1.3\varepsilon}{1-\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}} \sin\left(2 \arcsin \frac{1.3\varepsilon}{1-\varepsilon}\right)\right)$. Since $\varepsilon \leq \varepsilon_0$, the angle is at most $\arcsin \frac{\varepsilon_0\sqrt{3}}{1-\varepsilon_0} \approx 0.175$ radians. □

The following result is then a direct consequence of theorem 19 of [3].

Theorem 4.6 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then the restriction of the mapping π to $Del_{|S}(E)$ is a homeomorphism between $Del_{|S}(E)$ and S.*

Corollary 4.7 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then $Del_{|S}(E)$ and S are ambient isotopic.*

Proof Since the SMALL TRIANGLE CONDITION is verified by the facets of $\mathrm{Del}_{|S}(E)$, lemma 12 of [3] tells that $\forall x \in \mathrm{Del}_{|S}(E),\ \|x - \pi(x)\| < 0.165\ d_M(\pi(x))$. Moreover, according to theorem 4.6, π is a homeomorphism between $\mathrm{Del}_{|S}(E)$ and S. Thus, by theorem 9 of [5], $\mathrm{Del}_{|S}(E)$ and S are ambient isotopic. □

4.3 Hausdorff distance

Theorem 4.8 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then the Hausdorff distance between S and $Del_{|S}(E)$ is at most $8.5\,\varepsilon^2\ d_M^{\sup}$.*

The idea is to bound the distance from $\mathrm{Del}_{|S}(E)$ to S, and then to use the surjectivity of π to prove that the bound also holds for the distance from S to $\mathrm{Del}_{|S}(E)$.

Lemma 4.9 *Let $c \in S$. For any point $x \in S$ at distance at most $\varepsilon\ d_M(c)$ from c, the distance from x to $T(c)$ is at most $\frac{1}{2}\varepsilon^2 d_M(c)$.*

Proof Let B_1 and B_2 be the two balls of radius $d_M(c)$, tangent to S at c. Their interiors cannot intersect S and therefore do not contain x. Let x' be the intersection point other than c of the segment $[c, x]$ with the boundary of $B_1 \cup B_2$. Let h be the distance of x to $T(c)$ and θ the angle between $\overrightarrow{cx}$ and $T(c)$. We have

$$\|c - x'\| = 2d_M(c)\sin\theta \leq \|c - x\| \leq \varepsilon d_M(c)$$

Therefore, $\sin\theta \leq \frac{\varepsilon}{2}$ and $h = \|c - x\|\sin\theta \leq \frac{1}{2}\varepsilon^2 d_M(c)$. □

Lemma 4.10 *Let $c \in S$ and let y be a point of $T(c)$ at distance at most $\varepsilon\ d_M(c)$ from c. The distance of y to S is at most $8\varepsilon^2\ d_M(c)$.*

Proof Let z be the point of S closest to y, t its projection onto $T(c)$ and $\phi = \angle yzt$, which is also the angle between the normals to S at c and z. We have

$$\|c - z\| \leq \|c - y\| + \|y - z\| \leq 2\|c - y\| \leq 2\varepsilon\ d_M(c)$$

It then follows from lemma 4.9 that $\|z - t\| \leq 4\varepsilon^2\ d_M(c)$. Moreover, $d_M(c) \leq d_M(z) + \|c - z\|$, thus $\|c - z\| \leq \frac{2\varepsilon}{1-2\varepsilon} d_M(z)$. It follows from lemma 3.3 that $\phi \leq \frac{2\varepsilon}{1-8\varepsilon}$. Since $\varepsilon \leq \varepsilon_0 \leq 0.1$, we have $\frac{2\varepsilon}{1-8\varepsilon} \leq 1$, thus $\phi \leq 1$. It follows that $\frac{1}{\cos\phi} \leq \frac{1}{1-\frac{\phi^2}{2}} \leq 1 + \phi^2$, from which we deduce

$$\begin{aligned} \|y - z\| = \tfrac{\|z-t\|}{\cos\phi} &\leq 4\varepsilon^2\ d_M(c)\ \left(1 + \left(\tfrac{2\varepsilon}{1-8\varepsilon}\right)^2\right) \\ &\leq 8\varepsilon^2\ d_M(c) \end{aligned}$$

□

With lemmas 4.9 and 4.10, we can bound the distance from $\mathrm{Del}_{|S}(E)$ to S.

Proposition 4.11 *Every point $x \in Del_{|S}(E)$ is at distance at most $8.5\,\varepsilon^2 d_M(c) \leq 8.5\,\varepsilon^2 d_M^{\sup}$ from S, where c is the center of the surface Delaunay ball of the facet that contains x.*

Proof Let $x \in \mathrm{Del}_{|S}(E)$. Let f be a facet of $\mathrm{Del}_{|S}(E)$ on which x lies, and let $B(c, r)$ be the surface Delaunay ball of f. Let x' be the orthogonal projection of x onto $T(c)$. For any vertex a of f, we have $\|a - c\| \le r$, which is at most $\varepsilon\ d_M(c)$ since E is a loose ε-sample. Thus, by lemma 4.9, the distance from a to $T(c)$ is at most $\frac{1}{2}\varepsilon^2\ d_M(c)$. Since this is true for every vertex of f, it is also true for any point of f, and for x in particular. Hence, $\|x - x'\| \le \frac{1}{2}\varepsilon^2\ d_M(c)$. In addition, we have $\|x' - c\| \le \|x - c\| \le \varepsilon\ d_M(c)$. Thus, by lemma 4.10, the distance from x' to S is at most $8\varepsilon^2\ d_M(c)$. It follows that the distance from x to S is at most $8.5\ \varepsilon^2\ d_M(c) \le 8.5\ \varepsilon^2\ d_M^{\sup}$. □

We can now bound the distance from S to $\mathrm{Del}_{|S}(E)$, which completes the proof of theorem 4.8.

Proposition 4.12 *Every point $x \in S$ is at distance at most* $\min\{8.5\ \varepsilon^2 d_M^{\sup}, 10.5\ \varepsilon^2 d_M(x)\}$ *from* $\mathrm{Del}_{|S}(E)$.

Proof Let $x \in S$. Since the restriction of π to $\mathrm{Del}_{|S}(E)$ is surjective, we have $\pi^{-1}_{|\mathrm{Del}_{|S}(E)}(x) \neq \emptyset$. Let $x' \in \pi^{-1}_{|\mathrm{Del}_{|S}(E)}(x)$. According to proposition 4.11, $\|x - x'\| \le 8.5\ \varepsilon^2 d_M(c) \le 8.5\ \varepsilon^2 d_M^{\sup}$, where c is the center of the surface Delaunay ball of the facet that contains x'.

In addition, we have $\|x' - c\| \le \varepsilon\ d_M(c)$, since E is a loose ε-sample. Thus, $\|x - c\| \le (\varepsilon + 8.5\ \varepsilon^2) d_M(c) \le (\varepsilon + 8.5\ \varepsilon^2)(d_M(x) + \|x - c\|)$. It follows that $\|x - c\| \le \frac{\varepsilon + 8.5\ \varepsilon^2}{1 - \varepsilon - 8.5\ \varepsilon^2} d_M(x)$, which is at most $0.2\ d_M(x)$ since $\varepsilon \le \varepsilon_0$. Hence,

$$\|x - x'\| \le 8.5\ \varepsilon^2 d_M(c) \le 8.5\ \varepsilon^2 (d_M(x) + \|x - c\|) \le 10.5\ \varepsilon^2 d_M(x)$$

□

By lemma 2.3, we know that ε-samples are loose ε-samples. The converse is not true but the following corollary shows that loose ε-samples are close to be ε-samples.

Corollary 4.13 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \le \varepsilon_0 \approx 0.091$, such that $\mathrm{Del}_{|S}(E)$ has a vertex on each connected component of S. Then E is an $\varepsilon(1 + 16\ \varepsilon)$-sample of S.*

Proof By proposition 4.12, any point $x \in S$ is at distance at most $10.5\ \varepsilon^2\ d_M(x)$ from $\mathrm{Del}_{|S}(E)$. Let x' be the point of $\mathrm{Del}_{|S}(E)$ closest to x, and f the facet of $\mathrm{Del}_{|S}(E)$ that contains x'. We call c the center of the surface Delaunay ball of f, and c' the center of the circumcircle of f. Let a be the vertex of f closest to x'. Since x' belongs to f, we have $\|x' - a\| \le \|c' - a\| \le \|c - a\|$. Moreover, $\|c - a\| \le \varepsilon\ d_M(c) \le \varepsilon(d_M(a) + \|c - a\|)$, that is, $\|c - a\| \le \frac{\varepsilon}{1-\varepsilon} d_M(a)$. Thus,

$$\begin{aligned} \|x - a\| &\le \|x - x'\| + \|x' - a\| \\ &\le 10.5\ \varepsilon^2\ d_M(x) + \tfrac{\varepsilon}{1-\varepsilon}\ d_M(a) \\ &\le 10.5\ \varepsilon^2\ d_M(x) + \tfrac{\varepsilon}{1-\varepsilon}\ (d_M(x) + \|x - a\|) \end{aligned}$$

It follows that

$$\|x - a\| \le \frac{10.5(1 - \varepsilon)}{1 - 2\varepsilon}\ \varepsilon^2\ d_M(x) + \frac{\varepsilon}{1 - 2\varepsilon} d_M(x)$$

Since $\varepsilon \le \varepsilon_0$, we have $\frac{10.5(1-\varepsilon)}{1-2\varepsilon} \le 12$ and $\frac{1}{1-2\varepsilon} \le 1 + 4\varepsilon$, thus,

$$\|x - a\| \le 12\ \varepsilon^2\ d_M(x) + \varepsilon(1 + 4\varepsilon)\ d_M(x)$$

□

4.4 Covering

Let $\bigcup_{f \in \mathrm{Del}_{|S}(E)} B_f$ (or $\bigcup_f B_f$, for short) denote the union of the surface Delaunay balls.

Let $f_0 = (a, b, c)$ be a facet of $\mathrm{Del}_{|S}(E)$. Our goal is to prove that $C_{f_0} \subset \mathrm{int}\left(\bigcup_f B_f\right)$. In fact, we will prove a slightly more precise result, stated as lemma 4.14.

Let $F(f_0)$ be the set of all facets of $\mathrm{Del}_{|S}(E)$ that are incident to f_0, except f_0. Since, by theorem 4.5, $\mathrm{Del}_{|S}(E)$ is a manifold without boundary, $F(f_0)$ contains one facet of $\mathrm{Del}_{|S}(E)$ incident to f_0 through each edge of f_0. We define $R(f_0)$ as the union of all surface Delaunay patches associated with facets of $F(f_0)$.

Lemma 4.14 $C_{f_0} \subseteq int(R(f_0) \cup D_{f_0})$.

Proof We call f_{ab}, f_{bc} and f_{ac} the three facets of $F(f_0)$ that are incident to f_0 through edges ab, bc and ac respectively. By proposition 3.9, arcs ab, bc and ac of C_{f_0} are included in $D_{f_{ab}}$, $D_{f_{bc}}$ and $D_{f_{ac}}$ respectively, and only their endpoints may lie on $C_{f_{ab}}$, $C_{f_{bc}}$ or $C_{f_{ac}}$. Thus, the three arcs are included in the interior of $R(f_0)$, except for their endpoints which may possibly lie on some boundary of $R(f_0)$. In addition, by the claim of the proof of proposition 4.4, each vertex of f_0 lies in the interior of the union of surface Delaunay patches associated with its incident restricted Delaunay facets. These facets belong to $F(f_0) \cup \{f_0\}$, thus each vertex of f_0 lies in $\mathrm{int}(R(f_0) \cup D_{f_0})$. It follows that $C_{f_0} \subseteq \mathrm{int}(R(f_0) \cup D_{f_0})$, which ends the proof of the lemma. □

Theorem 4.15 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$. If $\bigcup_f B_f$ intersects all the connected components of S, then it covers S.*

Proof By lemma 4.14, the union of surface Delaunay patches has no boundary. Thus, S does not intersect the boundary of $\bigcup_f B_f$. Moreover, since we assumed that all the connected components of S intersect $\bigcup_f B_f$, S cannot exit $\bigcup_f B_f$ without intersecting the boundary of $\bigcup_f B_f$. It follows that $S \subseteq \bigcup_f B_f$. □

Recall that our definition of $\mathrm{Del}_{|S}(E)$ excludes edges and vertices with no incident restricted Delaunay facet. Hence there might exist points of E that are not vertices of $\mathrm{Del}_{|S}(E)$. In fact, this cannot happen, as stated in the following corollary of theorem 4.15.

Corollary 4.16 *Let S be a smooth closed surface and let E be a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then every point of E is a vertex of $Del_{|S}(E)$.*

Proof Let v be a point of E. By theorem 4.15, $S \subseteq \bigcup_f B_f$. Hence, v belongs to the surface Delaunay ball $B_f = B(c, r)$ of some facet f of $\mathrm{Del}_{|S}(E)$. Since B_f is a Delaunay ball, v belongs to its boundary. If f is incident to v, then v is a vertex of $\mathrm{Del}_{|S}(E)$. Otherwise, B_f has more than three points of E on its boundary, which means that its center c is dual to one or more Delaunay tetrahedra, one of which at least is incident to v. Let f' be a facet of this tetrahedron that is incident to v. Since c is dual to the tetrahedron, $c \in f'^*$ and hence f'^* intersects S. It follows that $f' \in \mathrm{Del}_{|S}(E)$ and that v is a vertex of $\mathrm{Del}_{|S}(E)$. □

Theorem 4.15 also induces another version of corollary 4.13, stated as corollary 4.17. Notice that, although the result is asymptotically weaker, the constant is better for $\varepsilon \geq \frac{15-\sqrt{161}}{32} \approx 0.073$.

Corollary 4.17 *Let S be a smooth closed surface and E a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then E is a $\frac{2\varepsilon}{1-\varepsilon}$-sample of S.*

Proof Each connected component of S intersects $\bigcup_f B_f$ since it contains a vertex of $\mathrm{Del}_{|S}(E)$. Thus, according to theorem 4.15, $S \subseteq \bigcup_f B_f$. Then, for every point $x \in S$, there exists a facet $f_x \in \mathrm{Del}_{|S}(E)$ such that x lies in $B_{f_x} = B(c_{f_x}, r_{f_x})$. So, $r_{f_x} \leq \varepsilon\, d_M(c_{f_x}) \leq \varepsilon\,(d_M(x) + \|x - c_{f_x}\|) \leq \varepsilon\,(d_M(x) + r_{f_x})$. It follows that $r_{f_x} \leq \frac{\varepsilon}{1-\varepsilon}\, d_M(x)$. Let a be a vertex of f_x. Since a also lies in B_{f_x}, we have $\|x-a\| \leq 2\, r_{f_x} \leq \frac{2\varepsilon}{1-\varepsilon}\, d_M(x)$, which implies that $\mathrm{dist}_E(x) \leq \frac{2\varepsilon}{1-\varepsilon}\, d_M(x)$, since the vertices of f belong to E. As this is true for any point x of S, E is a $\frac{2\varepsilon}{1-\varepsilon}$-sample of S. □

5 Size of loose ε-samples

5.1 Lower bound

Erickson [19] has shown that $\Omega\left(\frac{\mu(S)}{\varepsilon^2}\right)$, with $\mu(S) = \int_S \frac{dx}{d_M^2(x)}$, is a lower bound on the number of points of any ε-sample of S, with $\varepsilon < \frac{1}{5}$. This bound holds for loose ε-samples as well, by corollary 4.13. However, in the following we rewrite Erickson's proof in the case of loose ε-samples directly and improve on the constant.

Theorem 5.1 *Let S be a smooth closed surface and let E be a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0 \approx 0.091$, such that $\mathrm{Del}_{|S}(E)$ has a vertex on each connected component of S. Then $|E| \geq 2 + \frac{2}{5\pi}\frac{\mu(S)}{\varepsilon^2}$.*

Proof By theorem 4.15, we have $S \subseteq \bigcup_{f\in \mathrm{Del}_{|S}(E)} B_f$. Thus,

$$\mu(S) = \int_S \frac{dx}{d_M^2(x)} \leq \sum_{f\in \mathrm{Del}_{|S}(E)} \int_{D_f} \frac{dx}{d_M^2(x)} \tag{9}$$

Moreover, since E is a loose ε-sample, we have $\forall f \in \mathrm{Del}_{|S}(E)$, $\forall x \in D_f$, $\|x - c_f\| \leq \varepsilon d_M(c_f)$. It follows that $\|x - c_f\| \leq \frac{\varepsilon}{1-\varepsilon} d_M(x)$ and that $d_M(x) \geq (1-\varepsilon)d_M(c_f)$, since d_M is 1-Lipschitz. Thus,

$$\forall f \in \mathrm{Del}_{|S}(E), \quad \int_{D_f} \frac{dx}{d_M^2(x)} \leq \frac{\mathrm{Area}(D_f)}{(1-\varepsilon)^2 d_M^2(c_f)} \tag{10}$$

Since $\forall x \in D_f$, $\|x - c_f\| \leq \varepsilon d_M(c_f)$ and $\|x - c_f\| \leq \frac{\varepsilon}{1-\varepsilon} d_M(x)$, by lemma 3.3 we have $\forall x \in D_f$, $(\overrightarrow{n}(x), \overrightarrow{n}(c_f)) \leq \frac{\varepsilon}{1-4\varepsilon}$, which is less than $\frac{\pi}{2}$ since $\varepsilon \leq \varepsilon_0 < \frac{\pi}{2+4\pi}$. Thus, by lemma 9.4, D_f is a terrain over $T(c_f)$, the plane tangent to S at c_f. We can then bound the area of D_f by projecting it orthogonally onto $T(c_f)$. Let us call proj the orthogonal projection onto $T(c_f)$. Since $\mathrm{proj}(D_f)$ is included in the disk of radius $\varepsilon d_M(c_f)$ centered at c_f, we have

$$\mathrm{Area}(D_f) \leq \frac{\mathrm{Area}(\mathrm{proj}(D_f))}{\min_{x\in D_f} \cos\left(\overrightarrow{n}(x), \overrightarrow{n}(c_f)\right)} \leq \frac{\pi\varepsilon^2 d_M^2(c_f)}{\cos\frac{\varepsilon}{1-4\varepsilon}} \tag{11}$$

It follows from (9), (10) and (11) that

$$\mu(S) = \int_S \frac{dx}{d_M^2(x)} \leq \frac{\pi\varepsilon^2}{(1-\varepsilon)^2 \cos\frac{\varepsilon}{1-4\varepsilon}}\ m$$

where m is the number of facets of $\mathrm{Del}_{|S}(E)$. According to theorem 4.5, $\mathrm{Del}_{|S}(E)$ is a manifold without boundary, thus the number of vertices of $\mathrm{Del}_{|S}(E)$ is $2 + \frac{m}{2}$, by Euler's formula. Hence, $|E| \geq 2 + \frac{m}{2} \geq 2 + \frac{1}{2\pi}(1-\varepsilon)^2 \cos\frac{\varepsilon}{1-4\varepsilon}\frac{\mu(S)}{\varepsilon^2}$, which is at least $2 + \frac{2}{5\pi}\frac{\mu(S)}{\varepsilon^2}$ since $\varepsilon \leq \varepsilon_0$. □

5.2 Upper bound

Since adding points to an ε-sample results in another ε-sample, we cannot hope for an upper bound on ε-samples without making some additional assumptions. The same observation can also be made for loose ε-samples since ε-samples are loose ε-samples by lemma 2.3. This motivates the following definition.

Definition 5.2 *A loose ε-sample E of S is said to be κ-sparse if $\forall x \in E,\ E \cap B(x, \kappa\ \varepsilon\ d_M(x)) = \{x\}$.*

In this section, we give an upper bound on κ-sparse loose ε-samples or loose (ε, κ)-samples for short. To this end, we first bound the size of loose ε-samples with respect to the local feature size of the sample (proposition 5.7).

Definition 5.3 *The* local feature size *at point $x \in S$, denoted by $\rho(x)$, is the radius of the smallest ball centered at x that contains at least two points of E.*

The local feature size was first introduced by Ruppert [24], who showed that ρ is 1-Lipschitz. Notice that, for every point $v \in E$, $\rho(v)$ is the distance from v to its nearest neighbour in E. Hence, definition 5.2 is equivalent to

$$\forall v \in E,\ \rho(v) \geq \kappa\ \varepsilon\ d_M(v) \tag{12}$$

For every point $v \in E$, we define B_v as the open ball $B(v, \rho(v)/2)$. Notice that the balls B_v are pairwise disjoint. We first prove two lemmas that will be useful to establish lemma 5.6. The latter is useful to bound the cardinality of E.

Lemma 5.4 *For every point $v \in E$, B_v is included in the Voronoi region of v.*

Proof Let w be any point of E. We have

$$\forall x \in B_v,\ \|v - x\| \leq \frac{\rho(v)}{2} \leq \frac{\|v - w\|}{2}$$

Thus,

$$\|w - x\| \geq \|v - w\| - \|v - x\| \geq \|v - x\|$$

$\square$

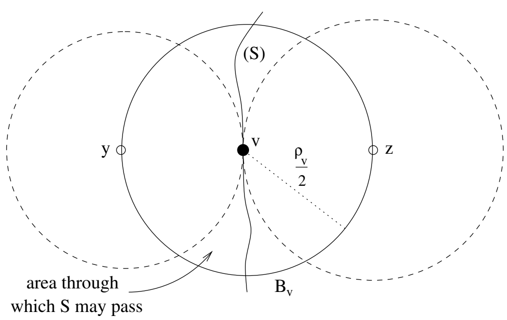
In the following, E is a loose ε-sample, with $\varepsilon \leq \varepsilon_0$, such that $\mathrm{Del}_{|S}(E)$ has a vertex on each connected component of S.

Lemma 5.5 *For every point $v \in E$, the radius $\frac{\rho(v)}{2}$ of B_v is at most $\frac{\varepsilon}{1-\varepsilon}\, d_M(v)$.*

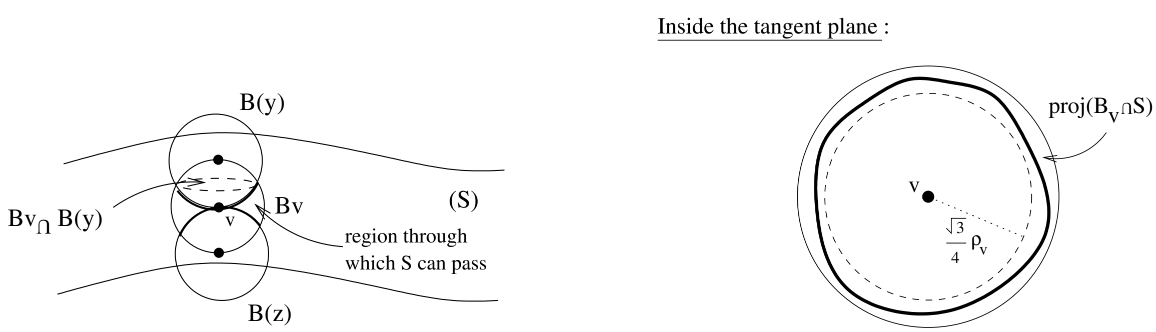
Proof Let v be a point of E. By corollary 4.16, v is a vertex of $\mathrm{Del}_{|S}(E)$, *i.e.* it is incident to some facet f of $\mathrm{Del}_{|S}(E)$. Let $B(c, r)$ be the surface Delaunay ball of f, and let w be another vertex of f. Since E is a loose ε-sample, we have $r \leq \varepsilon\, d_M(c) \leq \varepsilon(d_M(v) + r)$, *i.e.* $r \leq \frac{\varepsilon}{1-\varepsilon} d_M(v)$. Thus, $\|v - w\| \leq 2r \leq \frac{2\varepsilon}{1-\varepsilon} d_M(v)$, and $\frac{\rho(v)}{2} \leq \frac{\|v-w\|}{2} \leq \frac{\varepsilon}{1-\varepsilon} d_M(v)$. $\square$

Lemma 5.6 *For every point $v \in E$, we have $Area(S \cap B_v) \geq \frac{3}{16}\pi\rho^2(v)$.*

Proof According to lemma 5.5, $\frac{\rho(v)}{2} \leq \frac{\varepsilon}{1-\varepsilon} d_M(v)$, which is less than $d_M(v)$ since $\varepsilon < \frac{1}{2}$. Thus, $B_v \cap M = \emptyset$. It follows that $S \cap B_v$ is a topological disk, by lemma 3.6. It follows also that S does not intersect the open balls $B(y, \rho(v)/2)$ and $B(z, \rho(v)/2)$, where y and z are the intersection points of the line normal to S at v with the bounding sphere of B_v. Hence, $S \cap B_v$ lies outside these two balls.



Moreover, since S has no boundary, the boundary of $S \cap B_v$ lies in the bounding sphere of B_v. Thus, $\mathrm{proj}\,(S \cap B_v)$, the orthogonal projection of $S \cap B_v$ onto $T(v)$, contains the projection of $B_v \cap B\,(y, \rho(v)/2)$, which is a disk of radius $\frac{\sqrt{3}}{4}\rho(v)$.



Thus,

$$\begin{array}{rcl} \text{Area}(S \cap B_v) & \geq & \text{Area}\left(\text{proj}(S \cap B_v)\right) \\ & \geq & \frac{3}{16}\ \pi\ \rho^2(v) \end{array}$$

$\square$

Proposition 5.7 *Let S be a smooth closed surface and let E be a loose ε-sample of S, with $\varepsilon \leq \varepsilon_0$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then $|E| \leq \frac{12}{\pi}\ \int_S \frac{dx}{\rho^2(x)}$.*

Proof We proceed as in the case of planar meshes [18] and bound the integral of $1/\rho^2(x)$ over the whole surface. Since $S \cap B_v \subseteq S$ for every point $v \in E$, we have

$$\int_S \frac{dx}{\rho^2(x)} \geq \int_{\bigcup\limits_{v \in E} (B_v \cap S)} \frac{dx}{\rho^2(x)}$$

and since the balls B_v are pairwise disjoint,

$$\int_{\bigcup\limits_{v \in E} (B_v \cap S)} \frac{dx}{\rho^2(x)} = \sum_{v \in E} \int_{(B_v \cap S)} \frac{dx}{\rho^2(x)}$$

In addition, ρ is 1-Lipschitz, thus $\forall x \in B_v,\ \rho(x) \leq \rho(v) + \|x - v\| \leq \frac{3}{2}\ \rho(v)$. It follows that

$$\int_S \frac{dx}{\rho^2(x)} \geq \frac{4}{9} \sum_{v \in E} \frac{\text{Area}(B_v \cap S)}{\rho^2(v)}$$

Since $\text{Area}(B_v \cap S) \geq \frac{3}{16}\pi\rho^2(v)$, by lemma 5.6, we get

$$\int_S \frac{dx}{\rho^2(x)} \geq \frac{4}{9} \sum_{v \in E} \frac{3}{16}\pi \ = \ \frac{\pi}{12}\ |E|$$

$\square$

We can now establish an upper bound on the size of loose (ε, κ)-samples that matches the lower bound of theorem 5.1 for fixed κ.

Theorem 5.8 *Let S be a smooth closed surface, and let E be a loose (ε, κ)-sample of S, with $\varepsilon \leq \varepsilon_0$, such that $Del_{|S}(E)$ has a vertex on each connected component of S. Then $|E| \leq c\frac{\mu(S)}{\varepsilon^2}$ where c depends only on κ.*

Proof Since E is a loose (ε, κ)-sample, equation (12) holds for every point of E. If we can show that it holds for every point of S, with some constant κ' that depends only on κ, then the result will be proved, since, by proposition 5.7, we have

$$|E| \leq \frac{12}{\pi}\ \int_S \frac{dx}{\rho^2(x)} \leq \frac{12}{\pi}\ \int_S \frac{dx}{\kappa'^2 \varepsilon^2 d_M^2(x)} \leq \frac{12}{\pi \kappa'^2}\ \frac{\mu(S)}{\varepsilon^2}$$

Let $x \in S$ and let v be the nearest neighbour of x among the points of E. We have $\forall t \in E,\ \|x - v\| \leq \|x - t\|$, which gives $\|v - t\| \leq \|v - x\| + \|x - t\| \leq 2\|x - t\|$. This is true in particular for the nearest neighbour w of x among the points of $E \setminus \{v\}$. It follows that

$$\rho(x) = \|x - w\| \geq \frac{\|v - w\|}{2} \geq \frac{\rho(v)}{2}$$

which is at least $\frac{\kappa\varepsilon}{2} d_M(v)$, by (12). Thus,

$$\rho(x) \geq \frac{\kappa\varepsilon}{2} d_M(v) \geq \frac{\kappa\varepsilon}{2} \left(d_M(x) - \|x - v\|\right) \geq \frac{\kappa\varepsilon}{2} \left(d_M(x) - \rho(x)\right)$$

which implies $\rho(x) \geq \frac{\kappa}{2+\kappa\varepsilon}\ \varepsilon\ d_M(x) \geq \frac{\kappa}{2+\kappa/2}\ \varepsilon\ d_M(x)$. Taking $\kappa' = \frac{\kappa}{2+\kappa/2}$ completes the proof. $\square$

6 Application to surface sampling

In this section, we introduce our meshing algorithm and analyze its behaviour on smooth closed surfaces. Extensions of this algorithm and more experimental results can be found in [9] [1]

6.1 Algorithm

The algorithm is inspired from Chew's surface meshing algorithm [12]. It takes as input a pair (S, g), where S is a surface and g is a 1-Lipschitz function of space. The algorithm iteratively constructs a point sample $\bar{E}$, and maintains its restricted Delaunay triangulation $\mathrm{Del}_{|S}(\bar{E})$ throughout the process. We initialize $\bar{E}$ with a set E as described below. Procedure *insert(p)* adds point p to $\bar{E}$ and updates $\mathrm{Del}_{|S}(\bar{E})$. For a surface Delaunay ball B, *center*(B) returns the center of B. A surface Delaunay ball $B(c, r)$ is said to be *good* if $r \leq g(c)$, and *bad* otherwise.

```
INPUT: (S, g)
INITIALIZATION
   construct E and set Ē = E;
   compute Del|S(Ē)
REPEAT
   LET B be a bad surface Delaunay ball
   insert(center(B))
UNTIL all surface Delaunay balls are good
```

Upon termination, the algorithm returns $\bar{E}$ and $\mathrm{Del}_{|S}(\bar{E})$, *i.e.* a point sample and a simplicial mesh, and it is easily seen that all surface Delaunay balls are then good. The issue is to show that the algorithm indeed terminates.

6.2 Termination

Theorem 6.1 *If S is compact and if g is bounded from below by some positive constant h, then the algorithm terminates for any initial sample E of S.*

Proof For convenience we shall use the notion of *insertion radius*, defined below:

Definition 6.2 *Let v be a point inserted by the algorithm. We call* insertion radius *of v (denoted by r_v) the distance from v to the current point set $\bar{E}$, at the time when v is inserted. By convention, the insertion radius of a point v of the input point set E is the distance from v to $E \setminus \{v\}$.*

Let ρ_E be the smallest distance between two points of the initial sample E. For any point $v \in E$, we have $r_v \geq \rho_E$. At each step, the algorithm inserts the center of a bad surface Delaunay ball $B = B(c, r)$. Since B is a Delaunay ball, we have $r_c = r$ and, since B is bad, $r > g(c)$. Thus, $r_c > g(c) \geq h$. It follows that the insertion radius remains larger than $l = \min\{\rho_E, h\}$ throughout the course of the algorithm.

Hence, the points of $\bar{E}$ are centers of pairwise disjoint open balls of radius $l/2$, and, since S is compact, a packing argument shows that they are finitely many. The result follows. □

6.3 Construction of E

In order to apply our results (in particular theorems 4.6 and 4.8, and corollary 4.13), we need that $\mathrm{Del}_{|S}(\bar{E})$ has a vertex on each connected component of S. To guarantee this, we ask $\mathrm{Del}_{|S}(E)$ to have *seed-facets*, *i.e.* facets whose surface Delaunay balls $B(c, r)$ have radius at most $\frac{1}{3}g(c)$.

Lemma 6.3 *A seed-facet remains in $\mathrm{Del}_{|S}(\bar{E})$ throughout the course of the algorithm.*

[1] The results of this paper allow to improve the analysis given in [9].

Proof Let f_0 be a seed-facet of $\mathrm{Del}_{|S}(E)$. Assume that, in the course of the algorithm, f_0 stops being a facet of $\mathrm{Del}_{|S}(\bar{E})$. This implies that there exists a step at which the algorithm inserted a point v inside a surface Delaunay ball $B_{f_0} = B(c_{f_0}, r_{f_0})$ of f_0. By definition, v is the center of a surface Delaunay ball B_f of some facet f, such that the radius r_f of B_f is strictly greater than $g(v)$.
Since v lies inside B_{f_0}, we have $\|v - c_{f_0}\| < \frac{1}{3}\, g(c_{f_0})$. And, since g is 1-Lipschitz, $g(v) \geq g(c_{f_0}) - \|v - c_{f_0}\| > \frac{2}{3}\, g(c_{f_0})$. Let a be one of the vertices of f_0. Since a is in B_{f_0}, we have $\|a - v\| \leq 2\, r_{f_0} \leq \frac{2}{3}\, g(c_{f_0}) < g(v) < r_f$, which contradicts the fact that B_f is a Delaunay ball. $\square$

In practice, we take a first point x on S (e.g. a point of minimal abscissa) and then compute the points of E by intersecting S with random segments whose endpoints lie on a small sphere centered at x. The points of E are guaranteed to form at least one seed-facet. The process is repeated for each connected component of S.

6.4 Properties of $\bar{E}$

If function g is at most $\varepsilon\, d_M$, then the final point sample $\bar{E}$ is a loose ε-sample. If $\varepsilon \leq \varepsilon_0$, theorems 4.6, 4.8 and 4.15 and their corollaries apply.

Let us consider the case when function g equals $\varepsilon\, d_M$ and prove that the returned sample is then sparse. When g equals $\varepsilon\, d_M$, any v of $\bar{E} \setminus E$ is at distance at least $\varepsilon\, d_M(v)$ from $\bar{E}$ at the time when it is inserted. It follows that for any two points v and w of $\bar{E} \setminus E$, we have $\|v - w\| \geq \varepsilon\, d_M(v)$ or $\|v - w\| \geq \varepsilon\, d_M(w)$, depending on whether v was inserted last or not. In both cases, we have $\|v - w\| \leq \frac{\varepsilon}{1+\varepsilon} d_M(v)$. Similarly, if $v \in \bar{E} \setminus E$ and $w \in E$, we have $\rho(v) \geq \varepsilon\, d_M(v)$. Thus, for any $v \in \bar{E} \setminus E$, we have $\rho(v) \geq \frac{\varepsilon}{1+\varepsilon} d_M(v)$. Finally, with $\rho_E = \min_{v \in E} \rho(v)$, we conclude that for any $v \in \bar{E}$, $\rho(v) \geq \rho_{\min} = \min(\rho_E, \frac{\varepsilon}{1+\varepsilon} d_M(v))$. It follows that $\bar{E}$ is $\rho_{\min}$-sparse. Then, theorems 5.1 and 5.8 show that the number of points of $\bar{E}$ is within a constant factor of any loose ε-sample of S, for $\varepsilon \leq \varepsilon_0$.

6.5 Practicality of the algorithm

Our algorithm can be used in a wide variety of contexts, and we believe that it is a competitor to the celebrated marching cube algorithm [21, 11]. It can be applied to any surface provided that an oracle can check whether or not a given line segment intersects the surface and, in the affirmative, find an intersection point and the distance from that point to the medial axis of the surface. Hence we only need to know the surface at a finite number of points.

Although the algorithm is extremely simple, computing the distance to the medial axis may be a limitation in practice since the distance to the medial axis depends on the global shape of the surface. However, for some surfaces, e.g. skin surfaces [10], it is equal to the minimum radius of curvature and can therefore be estimated locally. Moreover, our algorithm works fine if, instead of taking $g = \varepsilon\, d_M$, we take $g = \varepsilon\, h$, where h is a positive function smaller than d_M. For example, we can set h to a constant $c \leq d_M^{\inf}$, where $d_M^{\inf}$ is the infimum of the distance of S to its medial axis. Such a constant may be known from the context or computed, as described below. If the algorithm is run with such a constant, then it is no longer guaranteed to produce a sparse loose ε-sample, but it still produces a loose ε-sample of size $O\left(\frac{1}{\varepsilon^2}\right)$ and a good approximation of the surface (see [9] for more details).

From such a sample, we can estimate the distance of a point $x \in S$ to the medial axis by the distance from x to its pole, i.e. the vertex of the Voronoi cell of x farthest from x (see [4, 7, 16]). In a second step, we can restart the meshing algorithm from the beginning, using a function g equal to ε times the estimated d_M. The result is a close-to-optimal loose ε-sample of S.

Finding a constant $c \leq d_M^{\inf}$ can be done by an iterative process. At each iteration we check whether a certain positive value r is at most $d_M^{\inf}$. If the surface is algebraic, this reduces to checking whether some algebraic system has a solution or not. The resultant of the system provides this information. If $r \leq d_M^{\inf}$, then we take $c = r$. Otherwise, we divide r by two and redo the comparison with $d_M^{\inf}$. Since $d_M^{\inf}$ is assumed to be strictly positive, we will find c in $O\left(\left|\log_2 d_M^{\inf}\right|\right)$ iterations. This method has the advantage to provide a reasonable constant c, lying between $\frac{1}{2} d_M^{\inf}$ and $d_M^{\inf}$.

7 Conclusion

We have introduced a new notion of surface sample, the so-called loose ε-samples. We have shown that loose ε-samples are $\varepsilon(1+16\varepsilon)$-samples and share the main properties of ε-samples. Checking if a sample E of a surface S is a loose ε-sample reduces to comparing the radii of the surface Delaunay balls with the distances of their centers to the medial axis of S. Hence we obtain a new sufficient condition for sampling a surface with topological and geometric properties. This condition is similar in spirit to other sampling conditions [17, 2, 3]. An important advantage of our condition is that it leads to a simple and provably correct algorithm to sample and mesh surfaces.

This paper has only considered the case of smooth closed surfaces. We plan to extend our work to surfaces with boundaries and to piecewise smooth surfaces. Observe that surfaces with isolated singular points or curves still admit loose ε-samples, while they do not admit ε-samples since the distance to the medial axis vanishes at a singular point. Experimental results [9] have shown that the algorithm is robust and can produce good geometric approximations of surfaces with singular points or curves.

Our approach can be used for curves in any dimension: extending the proofs of this paper is not difficult, and in fact the proofs are simpler. Further research is needed to extend this work to manifolds of codimension larger than one embedded in spaces of higher dimensions.

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Appendix: various lemmas

Theorem 9.1 *(Meusnier's theorem)*
Let S be a C^2 surface, m be a point of S, and ξ be a curve drawn on S that passes through m. ξ is parameterized by arclength. Then,

$$\|\xi''\| = \frac{I\!I(\xi', \xi')}{\cos\theta}$$

where ξ' and ξ'' are respectively the first and second derivatives of ξ at point m, $I\!I$ is the second fundamental form of S at m, and θ is the angle between the normal to S and the normal to ξ at m.

A proof of this theorem can be found in [6][2].

Lemma 9.2 *Let $\overrightarrow{v}$ be a vector of Euclidean space $\mathbb{R}^3$, and Π a vectorial plane that is not orthogonal to $\overrightarrow{v}$. Let $\overrightarrow{v^\pi}$ denote the orthogonal projection of $\overrightarrow{v}$ onto Π. For any vector $\overrightarrow{u}$ of $\Pi \setminus \{\overrightarrow{0}\}$, we have $(\overrightarrow{v}, \overrightarrow{u}) \geq \left(\overrightarrow{v}, \overrightarrow{v^\pi}\right)$.*

Proof Notice that $(\overrightarrow{v} - \overrightarrow{v^\pi})$ is orthogonal to Π. Thus, for all $\overrightarrow{x} \in \Pi$, $\overrightarrow{v}.\overrightarrow{x} = \overrightarrow{v^\pi}.\overrightarrow{x}$. Consider $C = \left\{\overrightarrow{x} \in \Pi \mid \|\overrightarrow{x}\| = \|\overrightarrow{v^\pi}\|\right\}$. For all $\overrightarrow{u} \in C$, we have $\overrightarrow{v^\pi}.\left(\overrightarrow{u} - \overrightarrow{v^\pi}\right) \leq 0$. Thus, $\overrightarrow{v}.\overrightarrow{u} = \overrightarrow{v}.\overrightarrow{v^\pi} + \overrightarrow{v}.\left(\overrightarrow{u} - \overrightarrow{v^\pi}\right) = \overrightarrow{v}.\overrightarrow{v^\pi} + \overrightarrow{v^\pi}.\left(\overrightarrow{u} - \overrightarrow{v^\pi}\right) \leq \overrightarrow{v}.\overrightarrow{v^\pi}$, which means that $\cos(\overrightarrow{v}, \overrightarrow{u}) \leq \cos(\overrightarrow{v}, \overrightarrow{v^\pi})$ since $\|\overrightarrow{u}\| = \|\overrightarrow{v^\pi}\|$. It follows that $(\overrightarrow{v}, \overrightarrow{u}) \geq (\overrightarrow{v}, \overrightarrow{v^\pi})$.

Now, let $\overrightarrow{u}$ be any non-zero vector of Π. There exists a unique vector $\overrightarrow{u^c} \in C$ such that $(\overrightarrow{u}, \overrightarrow{u^c}) = 0$. It follows that $(\overrightarrow{u}, \overrightarrow{v}) = (\overrightarrow{u^c}, \overrightarrow{v}) \geq (\overrightarrow{v}, \overrightarrow{v^\pi})$. ☐

[2] In the cited book, the right hand of the expression is wrong but corrected here.

Lemma 9.3 *Let f and g be two univariate functions of class C^2. Let x_a and x_b ($x_a < x_b$) be two reals such that*

$$\begin{cases} \text{(i)} \quad f(x_a) = g(x_a) \text{ and } f(x_b) = g(x_b) \\ \text{(ii)} \quad \forall x \in [x_a, x_b]\,,\ f(x) \geq g(x) \\ \text{(iii)}\ \forall x \in]x_a, x_b[\,,\ g''(x) \leq 0 \end{cases}$$

Then there exists a real $x_c \in]x_a, x_b[$ such that $f''(x_c) \leq g''(x_c) \leq 0$.

Proof By $f'(x_a)$ and $g'(x_a)$ we denote the right derivatives of f and g at x_a. Idem, by $f'(x_b)$ and $g'(x_b)$ we denote the left derivatives of f and g at x_b. (i) and (ii) imply that $f'(x_a) \geq g'(x_a)$, since otherwise there would exist a neigbourhood V_a of x_a such that $\forall x \in V_a \setminus \{x_a\}$, $\frac{f(x)-f(x_a)}{x-x_a} < \frac{g(x)-g(x_a)}{x-x_a}$, which would give that $f < g$ on $V_a \setminus \{x_a\}$, which contradicts (ii). Idem, we have $f'(x_b) \leq g'(x_b)$.

Now, Taylor-Lagrange formula (at first order), applied to function (f'-g'), tells that there exists a real $x_c \in]x_a, x_b[$ such that $(f'-g')'(x_c) = \frac{(f'-g')(x_b)-(f'-g')(x_a)}{x_b-x_a}$, which is negative since $f'(x_a) - g'(x_a) \geq 0$ and $f'(x_b) - g'(x_b) \leq 0$. It follows that $f''(x_c) - g''(x_c) \leq 0$. ☐

Lemma 9.4 *Let S be a closed compact surface embedded in $\mathbb{R}^3$, and let $\overrightarrow{v}$ be a vector. We choose an orthonormal frame (O, x, y, z) such that $\overrightarrow{v}$ is oriented along the $[0, z)$ direction. Let Ω be a convex subset of $\mathbb{R}^3$, such that $\forall x \in S \cap \Omega$, $(\overrightarrow{n}(x), \overrightarrow{v}) < \frac{\pi}{2}$, where $\overrightarrow{n}(x)$ denotes the normal to S at x. Then $S \cap \Omega$ is xy-monotone.*

Proof Since S is a closed compact surface, it divides $\mathbb{R}^3$ into two connected components: a compact volume, called V_{int}, and a non-compact volume, called V_{ext}. We can assume without loss of generality that S is oriented such that its normal always points towards V_{ext}. Let us assume for a contradiction that there exists a point (x_0, y_0) of plane (O, x, y) such that the vertical line d passing through (x_0, y_0) intersects $S \cap \Omega$ at least twice. Let (x_0, y_0, z_1) and (x_0, y_0, z_2) be two points of intersection that are consecutive along d. If there are not two such points, then this means that d intersects $S \cap \Omega$ along a segment (which is a degenerate case), and at each point of this segment the normal to S is orthogonal to d, and thus has a zero scalar product with $\overrightarrow{v}$, which contradicts the hypothesis of the lemma. So now we assume that points (x_0, y_0, z_1) and (x_0, y_0, z_2) do exist. By definition, they are consecutive among the points of $S \cap \Omega \cap d$. Since Ω is convex, $\Omega \cap d$ is a segment of d, hence points (x_0, y_0, z_1) and (x_0, y_0, z_2) are also consecutive among the points of $S \cap d$. Thus, the open segment of d that joins them is included in one component of $\mathbb{R}^3$: V_{int} or V_{ext}. Since the normal to S always points towards V_{ext}, it follows that $\overrightarrow{n}(x_0, y_0, z_1)$ or $\overrightarrow{n}(x_0, y_0, z_2)$ has a negative or zero scalar product with $\overrightarrow{v}$, which contradicts the hypothesis of the lemma. ☐

Lemma 9.5 *Let S be a closed compact surface embedded in $\mathbb{R}^3$, and let $\overrightarrow{v}$ be a vector. We choose an orthonormal frame (O, x, y, z) such that $\overrightarrow{v}$ is oriented along the $[0, z)$ direction. Let B be a ball centered at point $c \in S$, such that $\forall x \in B \cap S$, $(\overrightarrow{n}(x), \overrightarrow{v}) \leq \frac{\pi}{4}$. Then $S \cap B$ lies outside the cone K of apex c, of vertical axis and of half-angle $\frac{\pi}{4}$.*

Proof Let proj be the vertical projection onto plane (O, x, y). Since $B \cap S$ is xy-monotone by lemma 9.4, the projection of $B \cap S$ is one-to-one. Let us assume for a contradiction that there exists a point $c' \in B \cap S$ that lies inside K. Let P be the vertical plane that passes through c and c'. It intersects $B \cap S$ along a set of simple arcs, since $B \cap S$ is xy-monotone. We consider the segment that joins proj(c) and proj(c'):

1. if it is included in proj$(B \cap S)$, then c and c' belong to the same connected arc of $P \cap B \cap S$. The problem becomes then two dimensional: inside plane P, K is a cone of apex c, of vertical axis and of semi-angle $\frac{\pi}{4}$, and c and c' belong to a connected arc that is the graph of a function f, and whose normal makes an angle less that $\frac{\pi}{4}$ with the vertical direction (which means that $|f'| \leq 1$). Let (O, t, z) be an orthonormal frame of P. We call c_t and c'_t the t-coordinates of c and c'. We have $|f(c'_t) - f(c_t)| = \left|\int_{c_t}^{c'_t} f'(t)dt\right| \leq \left|\int_{c_t}^{c'_t} |f'(t)|dt\right| \leq |c'_t - c_t|$, which means that c' does not belong to K, which contradicts the assumption.

2. if the segment that joins proj(c) and proj(c') is not entirely included in proj$(B \cap S)$, then we have to find another point that satisfies all the assumptions of case 1. We call c'_1 and c'_2 the points of ∂B that have

same (x, y) coordinates as c', and we assume without loss of generality that c'_1 lies above c'_2. Let p_1 and p_2 be the upper and lower poles of ∂B. We consider the meridian m of ∂B than passes through p_1, c'_1, c'_2 and p_2. The smaller arc α_1 of m that joins p_1 and c'_1, and the smaller arc α_2 of m that joins p_2 and c'_2, project themselves onto the segment that joins $\text{proj}(c) = \text{proj}(p_1) = \text{proj}(p_2)$ and $\text{proj}(c') = \text{proj}(c'_1) = \text{proj}(c'_2)$. Since this segment is not entirely included in $\text{proj}(B \cap S)$, it intersects the boundary of $\text{proj}(B \cap S)$. Let c^* be the point of intersection that is closest to $\text{proj}(c)$. Since S is a closed surface, the boundary of $B \cap S$ belongs to ∂B, thus the point of $B \cap S$ that projects onto c^* lies on α_1 or α_2. We call $\text{proj}^{-1}(c^*)$ this point. Since $c' \in K$ by assumption, c'_1 and c'_2 also belong to K. Thus, all the points of α_1 belong to K since they are closer than c'_1 to the vertical axis of the cone, while their z-coordinate is bigger than that of c'_1. Idem, $\alpha_2 \subset K$. Thus, $\text{proj}^{-1}(c^*)$ belongs to K. Moreover, since c^* is the point of intersection that is closest to $\text{proj}(c)$, segment $[\text{proj}(c), c^*]$ is included in $\text{proj}(B \cap S)$. So, $\text{proj}^{-1}(c^*)$ verifies all the assumptions of case 1., and thus leads to the same contradiction. $\square$

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