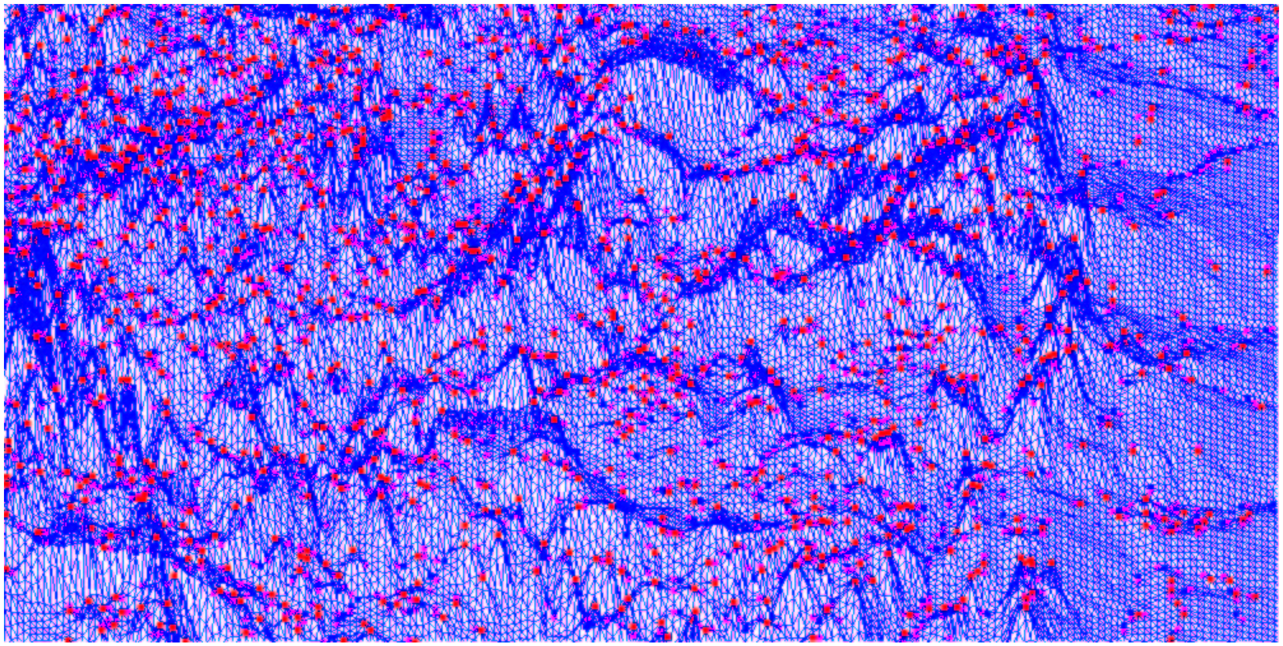
MORSE THEORY



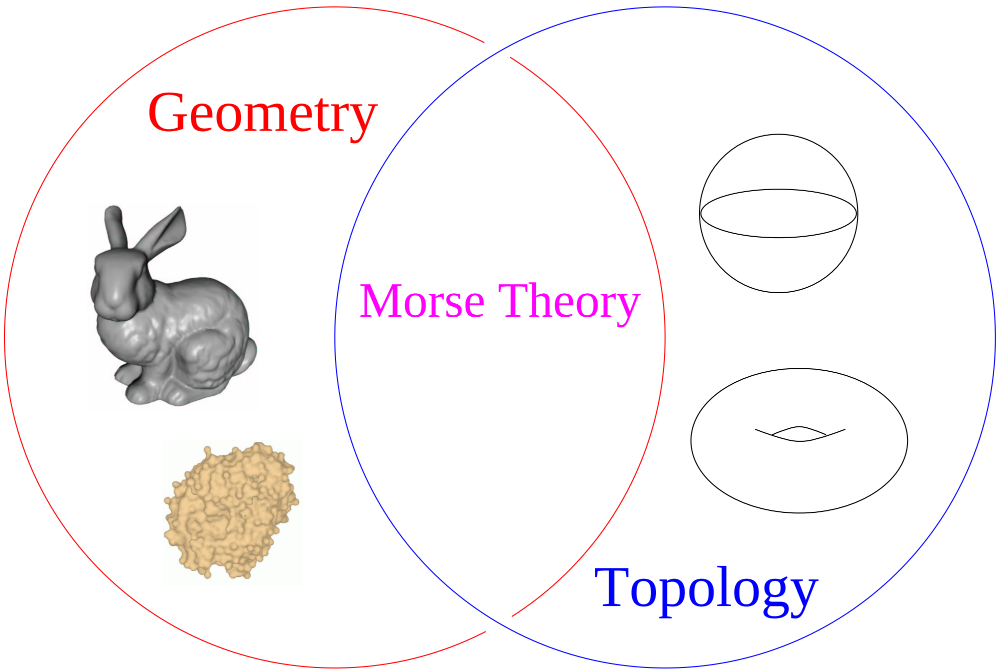
CS 468 – Lecture 9

11/20/2

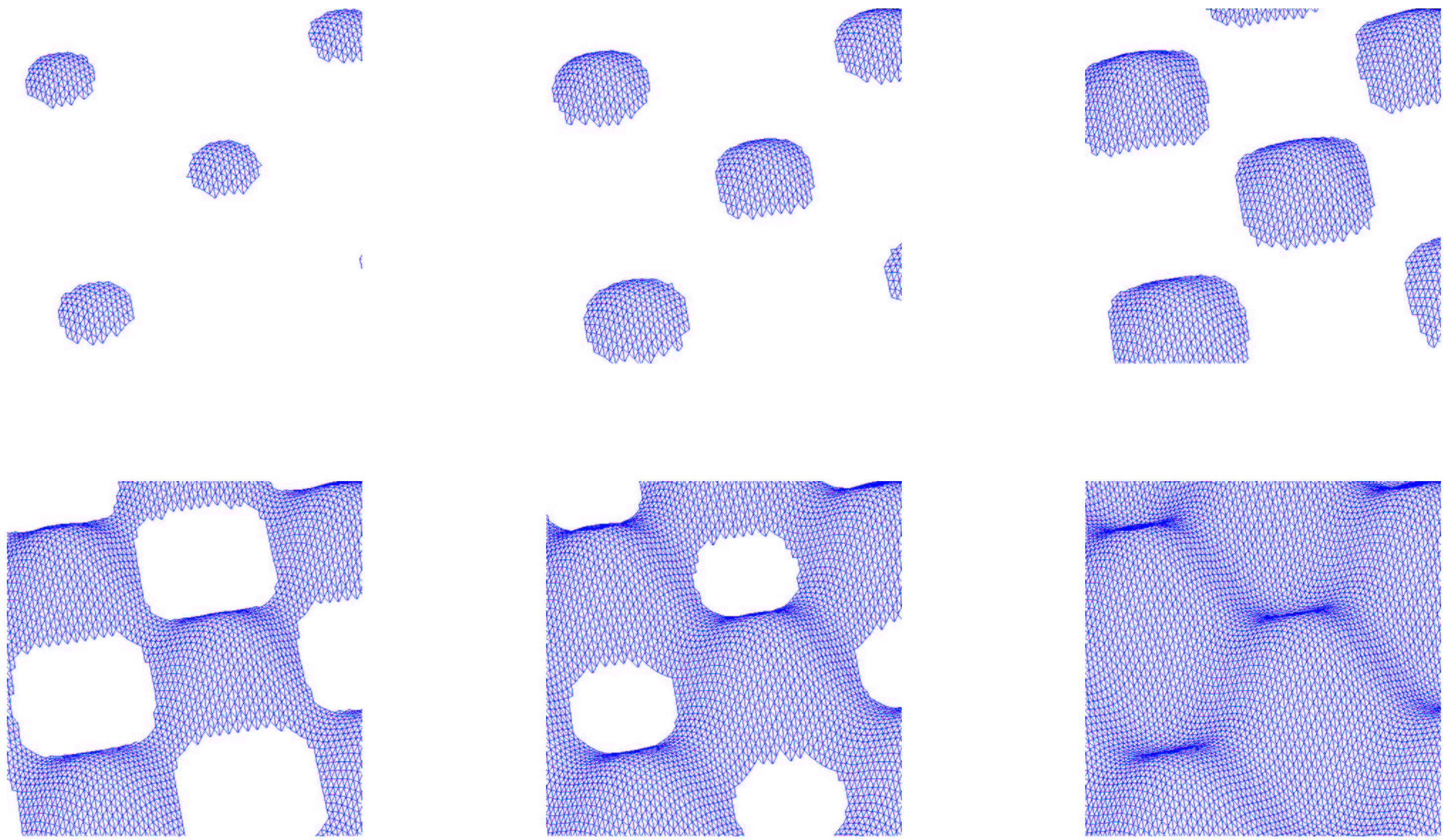
PRESENTATIONS

- November 27th:
  - Surface Flattening (Jie Gao)
  - Simplicial Sets (Patrick Perry)
  - Complexity of Knot Problems (Krishnaram Kenthapadi)
- December 4th
  - Tangent Complex (Yichi Gu)
  - Irreducible Triangulations (Jon McAlister)
  - Homotopy in the Plane (Rachel Kolodny)

SHAPE



EXCURSIONS



Overview

- Relationship between Geometry and Topology
- Tangent Spaces
- Derivatives
- Critical points
- Persistence

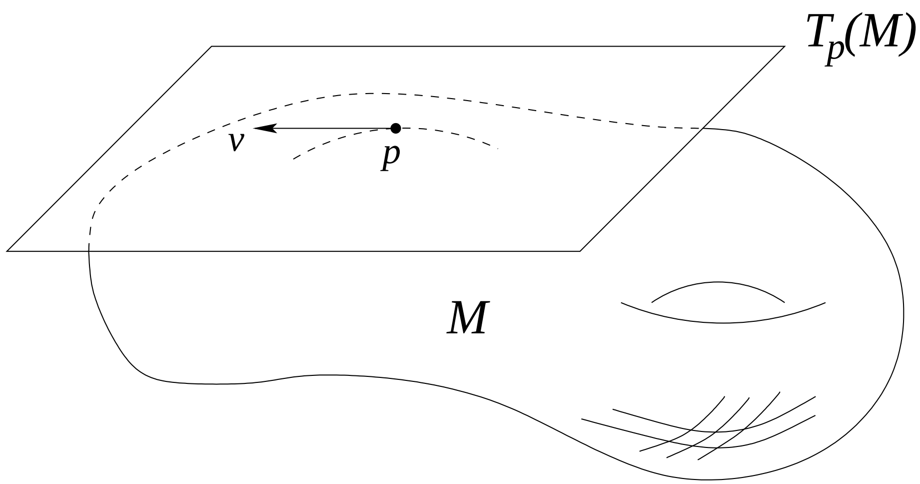
TANGENT SPACE $T_p(\mathbb{R}^3)$

- $\mathbb{M}$ is a smooth, compact, 2-manifold without boundary
- $\mathbb{M} \subset \mathbb{R}^3$ is embedded (not necessary, extends)
- A tangent vector v_p to $\mathbb{R}^3$ consists of two points of $\mathbb{R}^3$: its vector part v, and its point of application p.
- The set $T_p(\mathbb{R}^3)$ consists of all tangent vectors to $\mathbb{R}^3$ at p, and is called the tangent space of $\mathbb{R}^3$ at p.

TANGENTS SPACE $T_p(\mathbb{M})$

- Let p be a point on $\mathbb{M}$ in $\mathbb{R}^3$.
- A tangent vector v_p to $\mathbb{R}^3$ at p is tangent to $\mathbb{M}$ at p if v is the velocity of some curve in $\mathbb{M}$.
- The set of all tangent vectors to M at p is called the tangent plane of M at p and is denoted by $T_p(\mathbb{M})$.

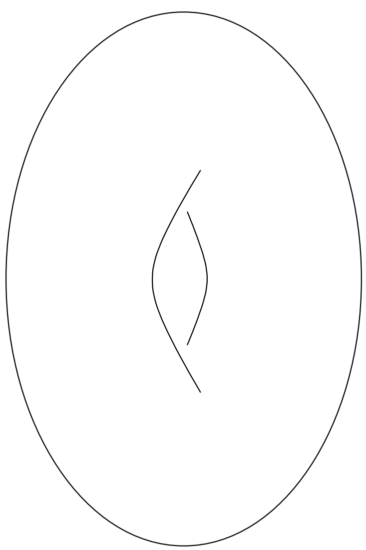
TANGENT PLANE



- A patch is the inverse of a chart.
- Let $p \in \mathbb{M} \subset \mathbb{R}^3$, and let φ be a patch in $\mathbb{M}$ such that $\varphi(u_0, v_0) = p$.
- A tangent vector v to $\mathbb{R}^3$ at p is tangent to $\mathbb{M}$ iff v can be written as a linear combination of $\varphi_u(u_0, v_0)$ and $\varphi_v(u_0, v_0)$.
- $T_p(\mathbb{M})$ is the best linear approximation of the surface M near p.

FUNCTIONS ON MANIFOLDS

- A vector: direction for moving
- Real valued smooth function $h : \mathbb{M} \rightarrow \mathbb{R}$.
- How does h vary in direction v_p?

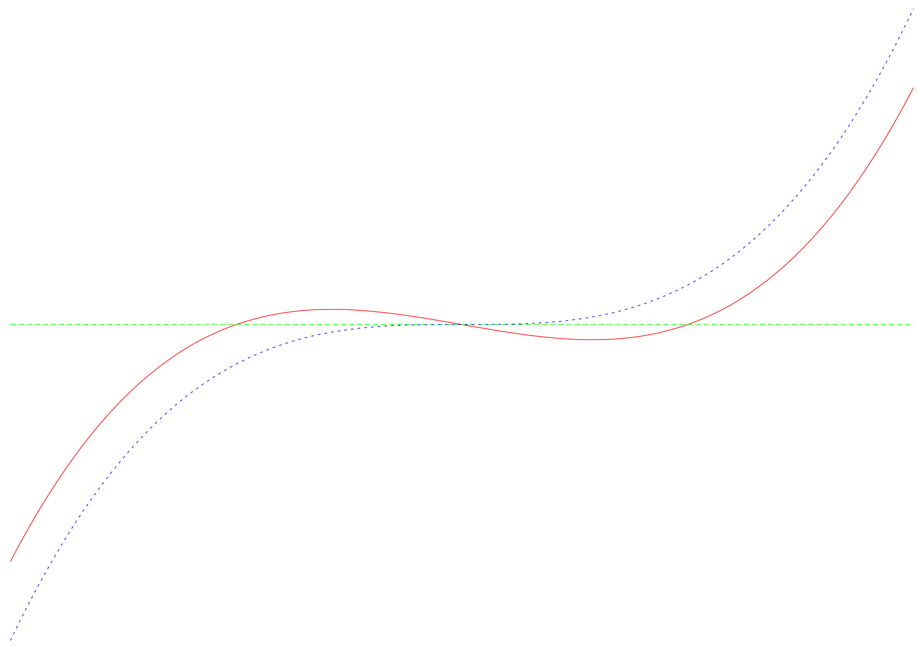


DERIVATIVES

- A vector field or flow on V is a function that assigns a vector $v_p \in T_p(\mathbb{M})$ to each point $p \in \mathbb{M}$.
- The derivative $v_p[h]$ of h with respect to v_p is the common value of $(d/dt)(h \circ \gamma)(0)$, for all curves $\gamma \in \mathbb{M}$ with initial velocity v_p.
- The differential dh_p of $h : \mathbb{M} \to \mathbb{R}$ at $p \in \mathbb{M}$ is a linear function on $T_p(\mathbb{M})$ such that $dh_p(v_p) = v_p[h]$, for all tangent vectors $v_p \in T_p(\mathbb{M})$.
- A differential converts vector fields to real-valued functions

CRITICAL POINTS

- Travel in all directions in $T_p(\mathbb{M})$
- A point $p \in \mathbb{M}$ is critical for map $h : \mathbb{M} \to \mathbb{R}$ if dh_p is the zero map.
- Otherwise, p is regular.



DEGENERACIES

- Let x, y be a patch on $\mathbb{M}$ at p.
- The Hessian of $h : \mathbb{M} \to \mathbb{R}$ is:

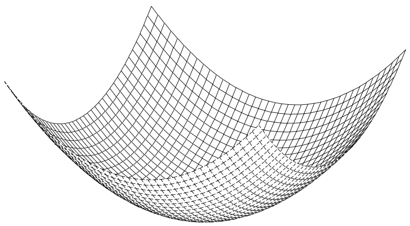
$$H(p) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(p) & \frac{\partial^2 h}{\partial y \partial x}(p) \\ \frac{\partial^2 h}{\partial x \partial y}(p) & \frac{\partial^2 h}{\partial y^2}(p) \end{bmatrix}.$$

- Basis $(\frac{\partial}{\partial x}(p), \frac{\partial}{\partial y}(p))$ for $T_p(\mathbb{M})$.
- A critical point $p \in \mathbb{M}$ is non-degenerate if the Hessian is non-singular at p, i.e. $\det H(p) \neq 0$.
- Otherwise, it is degenerate.

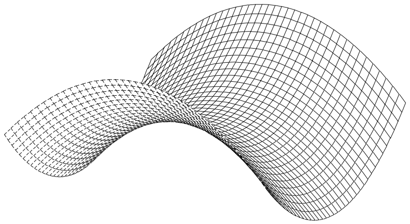
MORSE FUNCTIONS

- A smooth map $h : \mathbb{M} \rightarrow \mathbb{R}$ is a Morse function if all its critical points are non-degenerate.
- Any twice differentiable function h may be unfolded to a Morse function.
- As close to h as we specify!
- Morse functions are dense

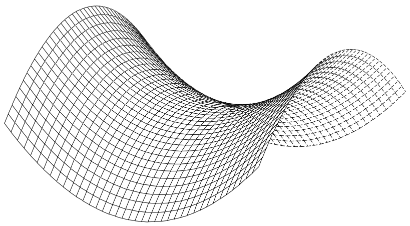
Morse Lemma



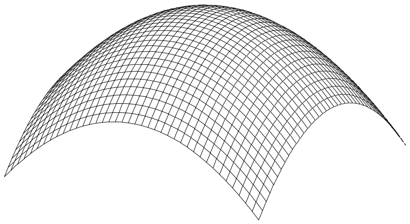
(a) $x^2 + y^2$



(b) $-x^2 + y^2$



(c) $x^2 - y^2$



(d) $-x^2 - y^2$

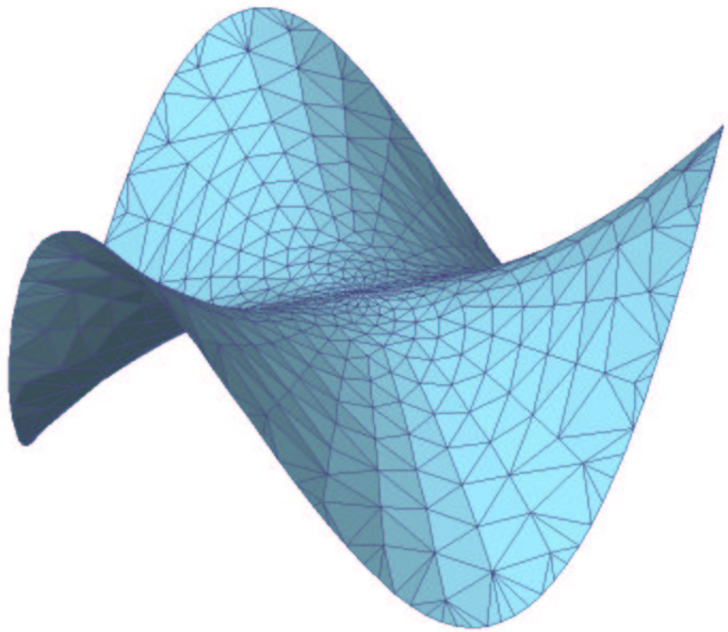
Indices

- (Lemma) It is possible to choose local coordinates x, y at a critical point $p \in \mathbb{M}$, so that a Morse function h takes the form:

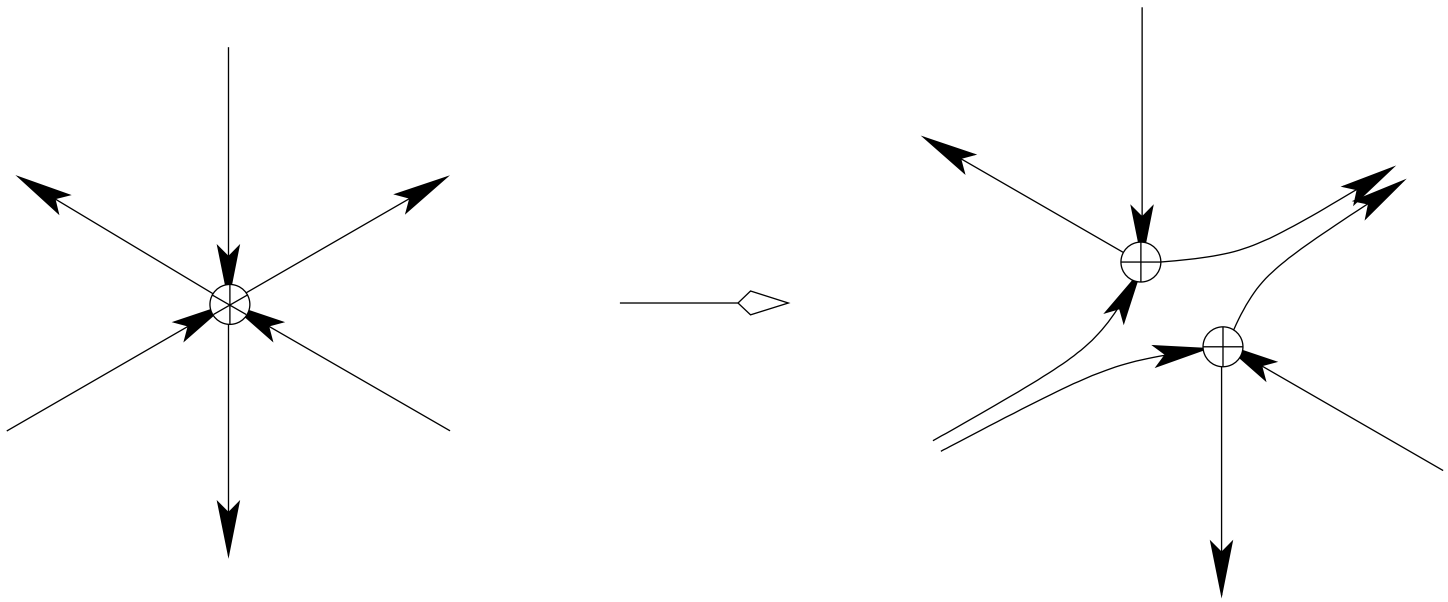
$$h(x, y) = \pm x^2 \pm y^2 \qquad (1)$$

- The index i(p) of h at critical point $p \in \mathbb{M}$ is the number of minuses.
- Equivalently, the index at p is the number of the negative eigenvalues of $H(p)$.
- A critical point of index 0, 1, or 2, is called a minimum, saddle, or maximum, respectively.

MONKEY SADDLE



UNFOLDING



PL FUNCTIONS

- Let K be a triangulation of a compact 2-manifold without boundary $\mathbb{M}$.
- Let $h : \mathbb{M} \to \mathbb{R}$ be a function that is linear on every triangle.
- The function is defined by its values at the vertices of K.
- Assume $h(u) \neq h(v)$ for all vertices $u \neq v \in K$.
- Sometimes called a height function over a 2-manifold.

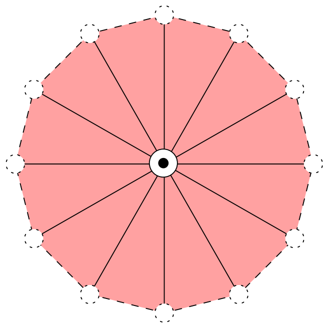
STARS

- Recall: the star of a vertex u in a triangulation K is $\text{St}\, u = \{\sigma \in K \mid u \leq \sigma\}$.
- The lower and upper stars of u for a height function h are

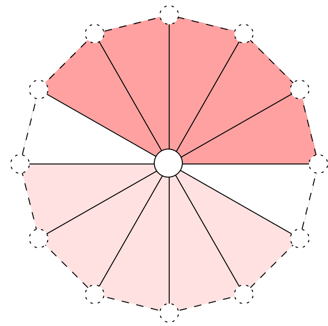
$$\begin{aligned} \underline{\text{St}}\, u &= \{\sigma \in \text{St}\, u \mid h(v) \leq h(u), \forall \text{ vertices } v \leq \sigma\} \\ \overline{\text{St}}\, u &= \{\sigma \in \text{St}\, u \mid h(v) \geq h(u), \forall \text{ vertices } v \leq \sigma\} \end{aligned}$$

- Suppose u is a maximum. What's $\underline{\text{St}}\, u$? What's $\overline{\text{St}}\, u$?
- $K = \dot{\bigcup}_u \underline{\text{St}}\, u = \dot{\bigcup}_u \overline{\text{St}}\, u$.

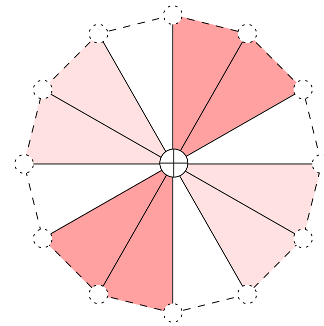
PL STARS



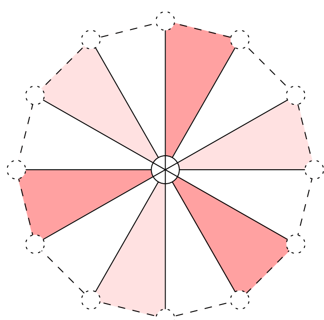
(a) minimum



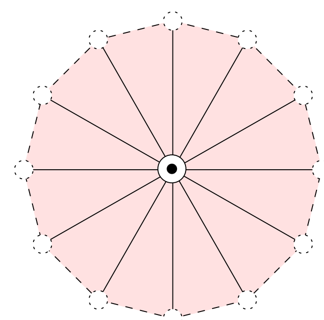
(b) regular



(c) saddle



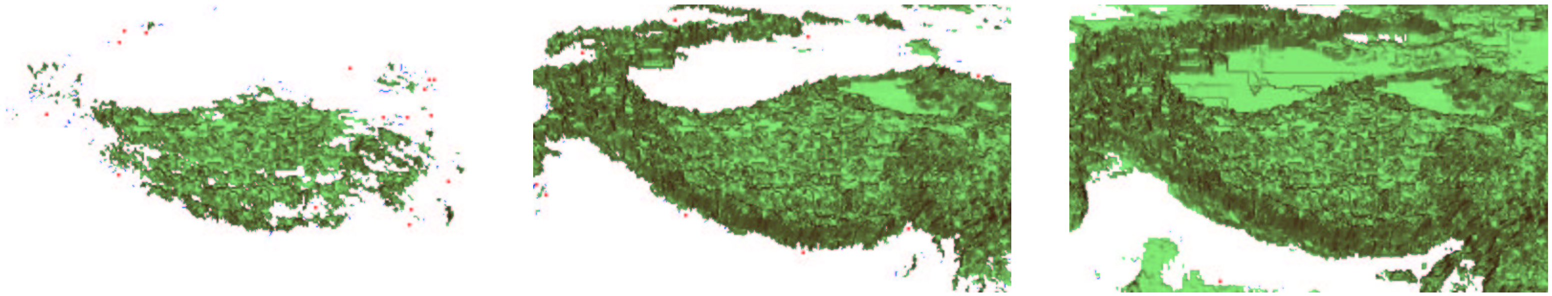
(d) monkey



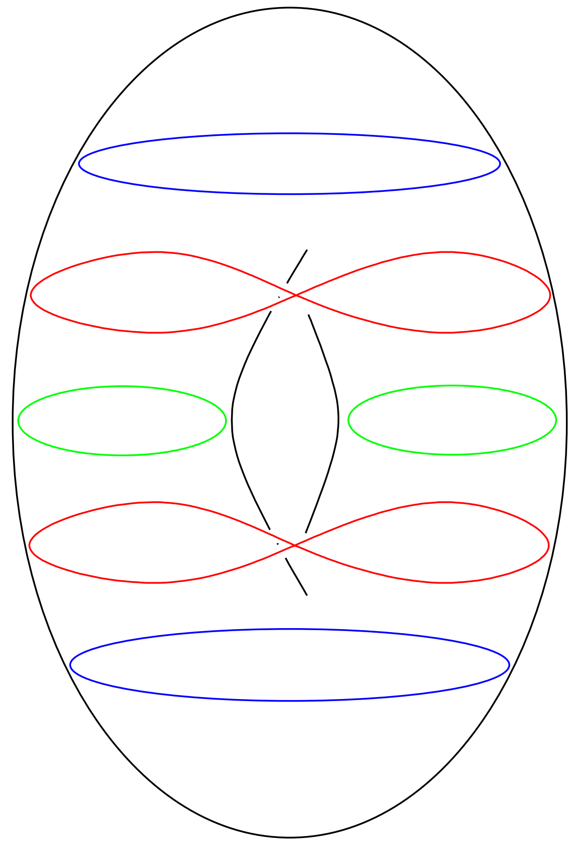
(e) maximum

FILTRATIONS

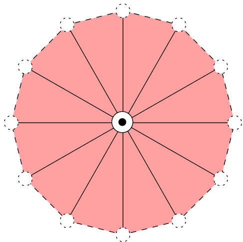
- Sort the n vertices of K in order of increasing height to get the sequence $u^1, u^2, \ldots, u^n$, $h(u^i) < h(u^j)$, for all $1 \leq i < j \leq n$.
- Let K^i be the union of the first i lower stars, $K^i = \bigcup_{1 \leq j \leq i} \underline{\mathrm{St}}\, u^j$.
- Same idea with upper stars
- Recall $\chi = v - e + t = \beta_0 - \beta_1 + \beta_2$



Levels of Torus

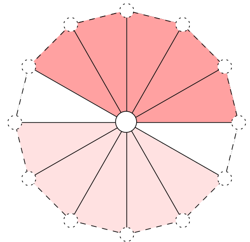


MINIMUM



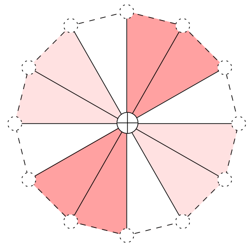
- $\underline{\text{St}}\, u^i = u^i$, so a minimum vertex is a new component and $\chi^i = \chi^{i-1} + 1$.
- $\beta_0^i = \beta_0^{i-1} + 1$, $\beta_1^i = \beta_1^{i-1}$, $\beta_2^i = \beta_2^{i-1}$
- Therefore, $\chi^i = \beta_0^{i-1} + 1 - \beta_1^{i-1} + \beta_2^{i-1} = \chi^{i-1} + 1$
- So, a minimum creates a new 0-cycle and acts like a positive vertex in the filtration of a complex.
- The vertex is unpaired at time i.

REGULAR



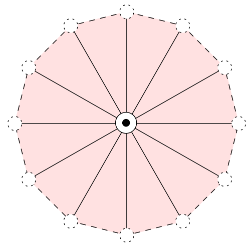
- $\underline{\text{St}}\, u^i$ is a single wedge, so $\chi^i = \chi^{i-1} + 1 - 1 = \chi^{i-1}$.
- $\underline{\text{St}}\, u^i \neq \emptyset$ so $\beta_0^i = \beta_0^{i-1}$
- $\overline{\text{St}}\, u^i \neq \emptyset$ so $\beta_2^i = \beta_2^{i-1}$ (also duality!)
- Using Euler-Poincaré, we get $\beta_1^i = \beta_1^{i-1}$.
- No topological changes!
- All the cycles created at time i are also destroyed at time i.
- The positive and negative simplices in $\underline{\text{St}}\, u^i$ cancel each other, leaving no unpaired simplices.

SADDLE



- $\underline{\mathrm{St}}\, u^i$ has two wedges, bringing in two more edges than triangles.
- $\chi^i = \chi^{i-1} + 1 - 2 = \chi^{i-1} - 1.$
- If this saddle connects two components, it destroys a 0-cycle and $\beta_0^i = \beta_0^{i-1} - 1.$
- Otherwise, it creates a new 1-cycle and $\beta_1^i = \beta_1^{i-1} + 1.$
- All the simplices in a saddle are paired, except for a single edge whose sign corresponds to the action of the saddle.
- We have $\chi^i = \chi^{i-1} - 1$ in either case.

MAXIMUM



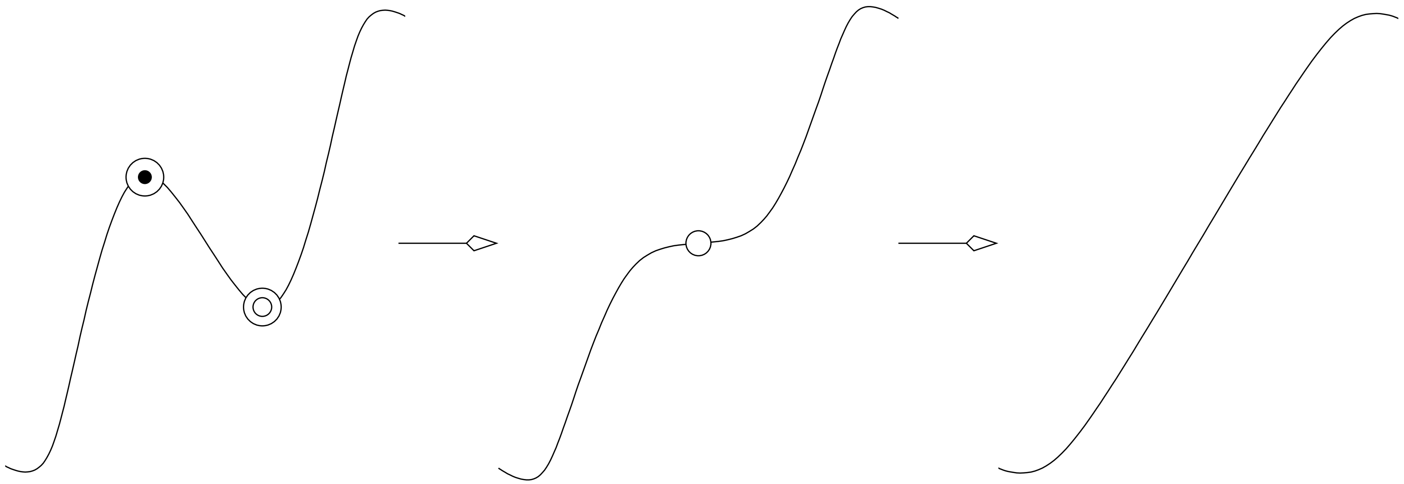
- $\underline{\mathrm{St}}\, u^i = \mathrm{St}\, u^i$ and has the same number of edges and triangles.
- So, $\chi^i = \chi^{i-1} + 1$ for the single vertex.
- In our case, one global minimum
- If global minimum, $\beta_2^i = \beta_2^{i-1} + 1 = 1$.
- Otherwise, the lower star covers a 1-cycle and $\beta_1^i = \beta_1^{i-1} - 1$.
- Single unpaired triangle (positive or negative)
- We have $\chi^i = \chi^{i-1} + 1$ in both cases.

CORRESPONDENCE

critical	unpaired	action
minimum	vertex	$\beta_{0^{++}}$
saddle	edge	$\beta_{0^{--}}$ or $\beta_{1^{++}}$
maximum	triangle	$\beta_{1^{--}}$ or $\beta_{2^{++}}$

- Correspondence allows us to talk about persistent critical points
- Let m_i be the number of index i critical points in K
- $\chi(K) = \sum_i (-1)^i s_i = \sum_i (-1)^i \beta_i = \sum_i (-1)^i m_i$

CANCELLATION



- Pairs of critical points annihilate each other
- Inverse unfolding plus smoothing
- Need additional structure (Morse-Smale Complex) to do this geometrically