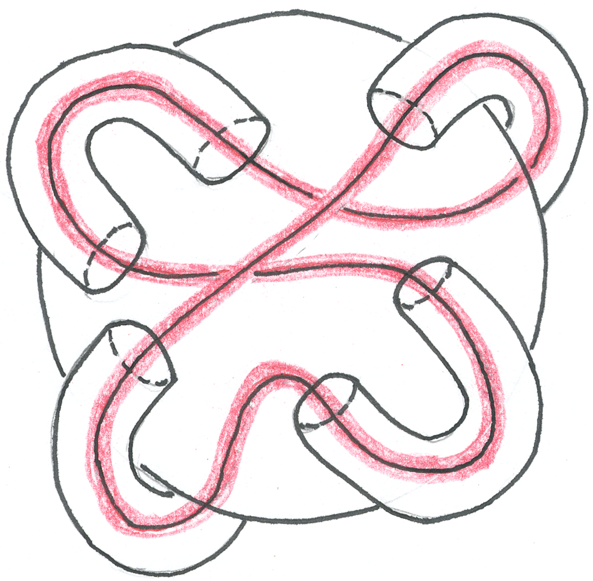
HOMOTOPY



CS 468 – Lecture 5

10/23/2

OVERVIEW

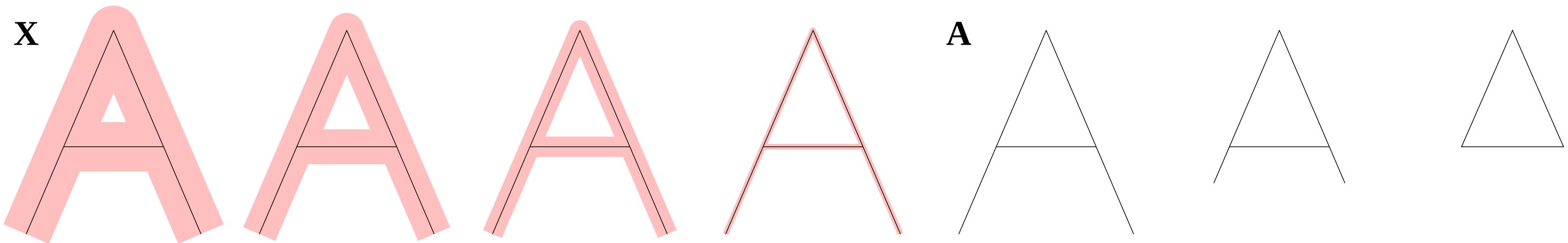
- Homotopy
- Categories and Functors
- Fundamental Group
- Markov's proof

DAVID HILBERT

Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the relationship of the ideas in mathematics as a whole, and the numerous analogies in its different departments.

— David Hilbert (1900)

Deformation Retraction

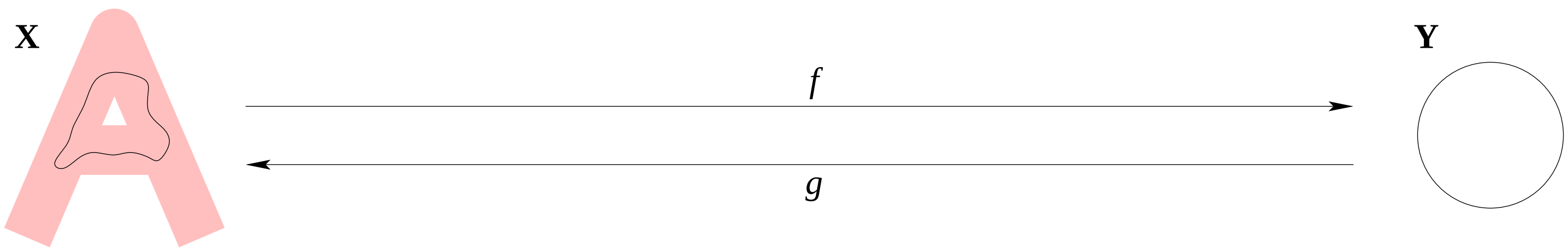


- Family of maps $f_t : \mathbb{X} \to \mathbb{X}, t \in [0,1]$
- f_0 is the identity map
- $f_1(\mathbb{X}) = \mathbb{A}$
- $f_t|\mathbb{A}$ is the identity map, for all t
- Family is continuous, i.e. $\mathbb{X} \times [0,1] \;\to\; \mathbb{X}$, $(x,t) \;\mapsto\; f_t(x)$ is continuous.
- f is a deformation retraction

HOMOTOPY

- Family of maps $f_t : \mathbb{X} \to \mathbb{Y}, t \in [0, 1]$
- Family is continuous, i.e. $\mathbb{X} \times [0, 1] \to \mathbb{Y}$, $(x, t) \mapsto f_t(x)$ is continuous.
- f is a homotopy
- f_0, f_1 are homotopic, $f_0 \simeq f_1$

HOMOTOPY EQUIVALENCE



- $f : \mathbb{X} \to \mathbb{Y}$, $g : \mathbb{Y} \to \mathbb{X}$
- $(f \circ g) \simeq 1_{\mathbb{X}}$, $(g \circ f) \simeq 1_{\mathbb{Y}}$
- $\mathbb{X}$ and $\mathbb{Y}$ are homotopy equivalent
- $\mathbb{X}$ and $\mathbb{Y}$ have the same homotopy type
- $\mathbb{X} \simeq \mathbb{Y}$
- $\simeq$ is an equivalence relation
- Contracible spaces $\simeq$ a point
- (Theorem) $\mathbb{X} \approx \mathbb{Y} \Rightarrow \mathbb{X} \simeq \mathbb{Y}$

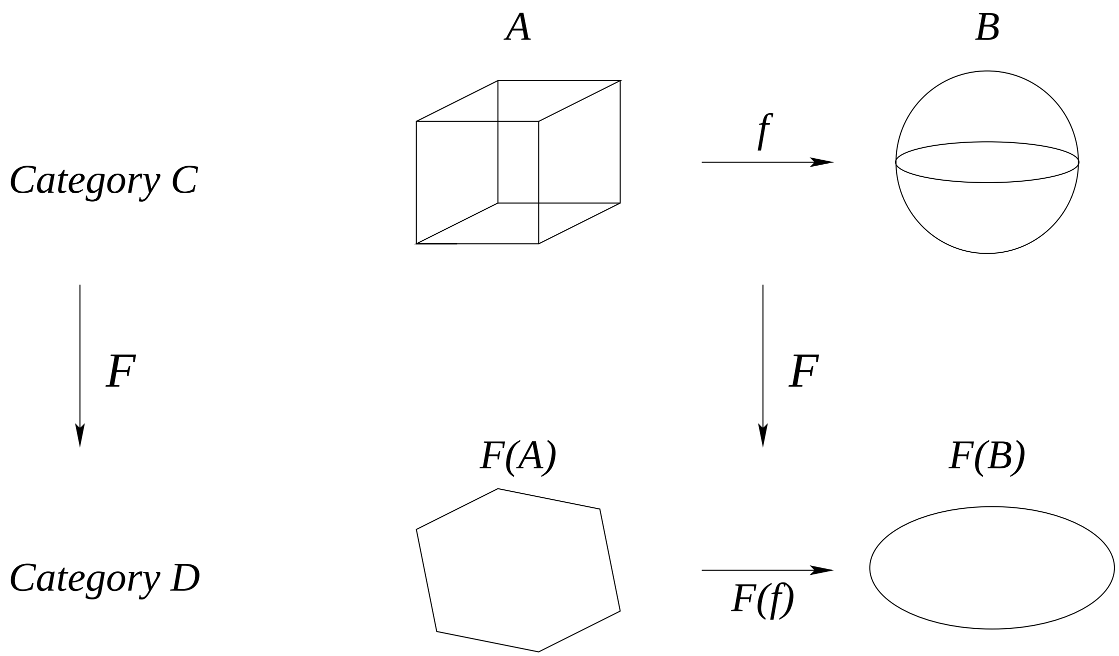
CATEGORIES

- A collection $\text{Ob}(\mathcal{C})$ of objects
- Sets $\text{Mor}(X, Y)$ of morphisms for each pair $X, Y \in \text{Ob}(\mathcal{C})$
- An identity morphism $1 = 1_X \in \text{Mor}(X, X)$ for each X.
- a composition of morphisms function $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$ for each triple $X, Y, Z \in \text{Ob}(\mathcal{C})$, satisfying $f \circ 1 = 1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.
- A category $\mathcal{C}$

SOME CATEGORIES

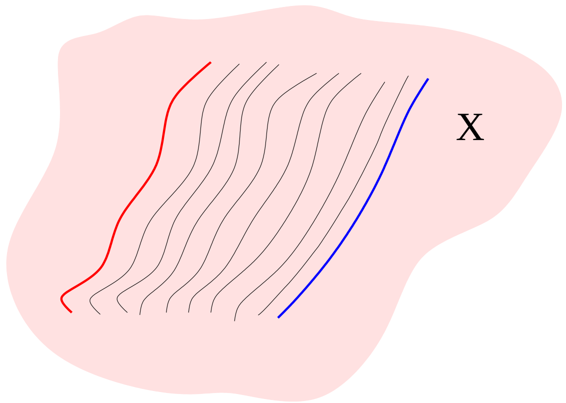
category	morphisms
sets	arbitrary functions
groups	homomorphisms
topological spaces	continuous maps
topological spaces	homotopy classes of maps

FUNCTORS



- $X \in \mathcal{C}$, $F(X) \in \mathcal{D}$,
- $f \in \mathrm{Mor}(X, Y)$, $F(f) \in \mathrm{Mor}(F(X), F(Y))$
- $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$
- F is a (covariant) functor

LOOPS



- A path in $\mathbb{X}$ is a continuous map $f : [0, 1] \to \mathbb{X}$.
- A loop is a path f with $f(0) = f(1)$, i.e. a loop starts and ends at the same base-point.
- The equivalence class of a path f under the equivalence relation of homotopy is $[f]$.

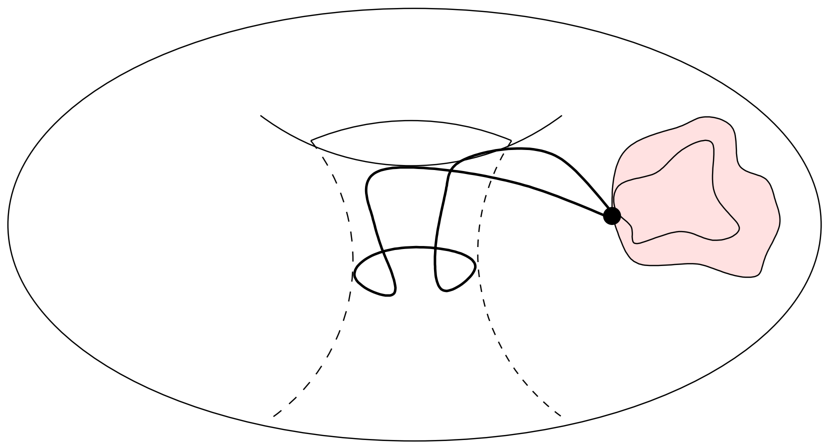
GROUPS (REMINDER)

- A group $\langle G, * \rangle$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:
  (a) $*$ is associative.
  (b) G has an identity e element for $*$ such that $e * x = x * e = x$ for all $x \in G$.
  (c) any element a has an inverse a' with respect to the operation $*$, i.e. $\forall a \in G, \exists a' \in G$ such that $a' * a = a * a' = e$.

FUNDAMENTAL GROUP

- Given two paths $f, g : [0, 1] \to \mathbb{X}$, the product path $f \cdot g$ is a path which traverses f and then g
- Over loops, the product is closed and associative
- Identity: the trivial loop is homotopy equivalent to a point
- Inverse: go backwards
- The fundamental group $\pi_1(\mathbb{X}, x_0)$ of $\mathbb{X}$ and x_0 has the homotopy classes of loops in $\mathbb{X}$ based at x_0 as its elements, and $[f][g] = [f \cdot g]$ as its binary operation

$\pi_1(\mathbb{T}^2)$



- A loop on manifold $\mathbb{M}$ that is the boundary of a disk is a boundary. Otherwise, the loop is non-bounding.
- All boundaries are contractible and homotopic to the trivial loop.
- $\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$

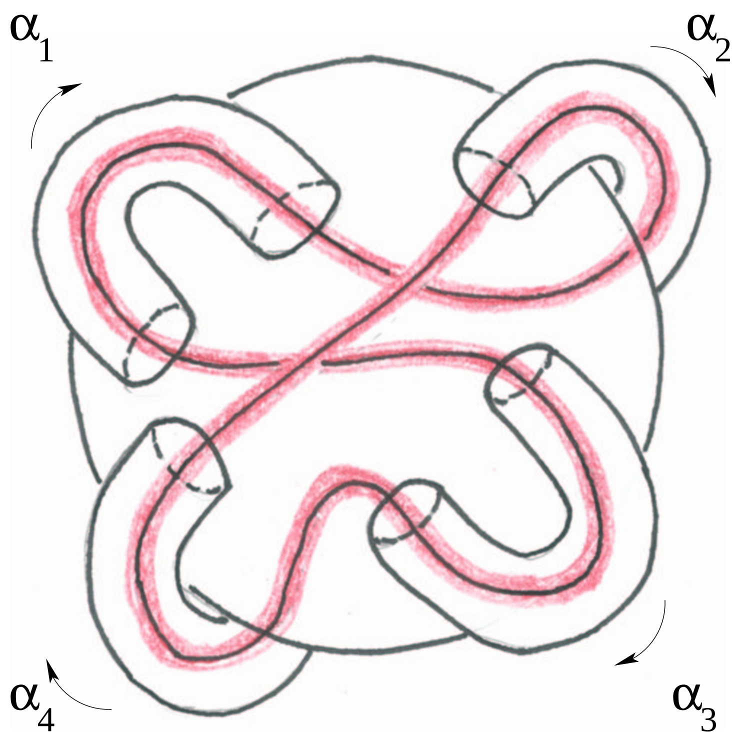
HOMOTOPY GROUPS

- $\pi_1(\mathbb{X})$ is the first of many homotopy groups $\pi_n(\mathbb{X})$ for a space $\mathbb{X}$.
- n-dimensional cycles
- $\mathbb{X} \approx \mathbb{Y}$ implies $\pi_n(\mathbb{X}) = \pi_n(\mathbb{Y})$, for all n
- Not the other way around
- Problems:
  1. not combinatorial
  2. very complicated, not directly computable from simplicial complexes
  3. maybe an infinite description of space

MARKOV'S PROOF

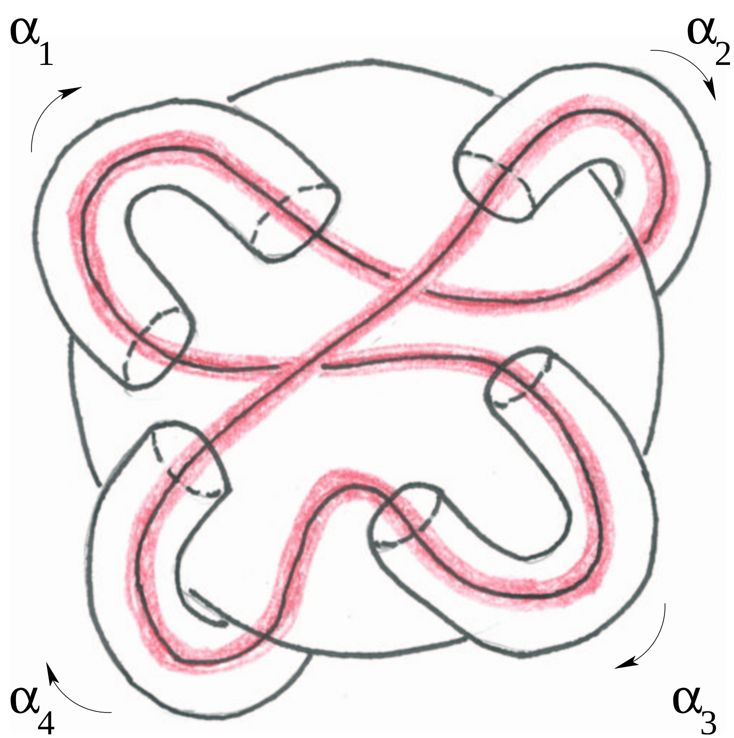
- Fundamental group is a functor from the category of topological spaces to the category of groups
- Reduce the homeomorphism problem to the isomorphism problem of groups
- (Dehn 1912) Given two finitely presented groups, decide whether or not they are isomorphic.
- (Adyan 1955) For any fixed group, Dehn's problem is undecidable.
- Given a finitely presented group $G : (a_1, \ldots, a_n : r_1, \ldots, r_m)$
- Build a manifold whose fundamental group is G!

GENERATORS



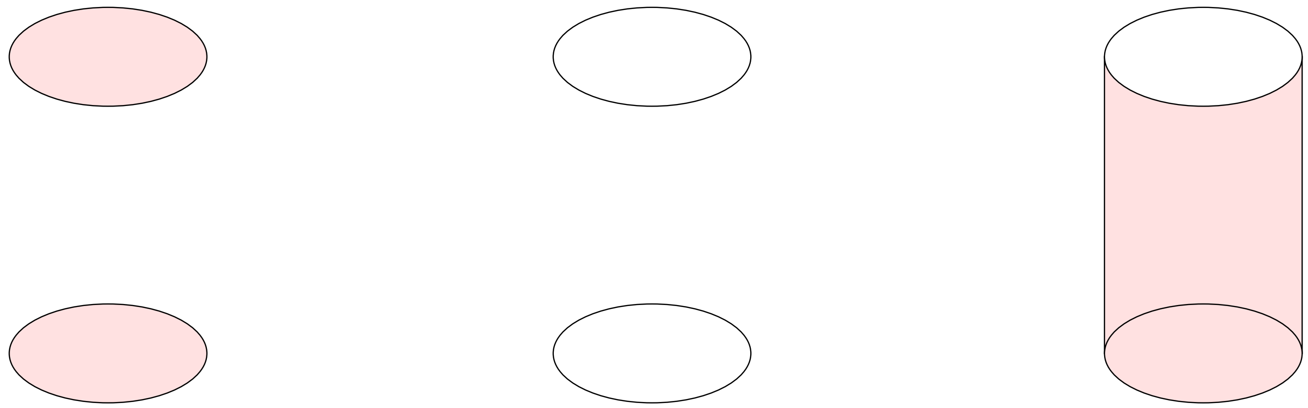
- Take a four-dimensional closed ball $\mathbb{B}^4$
- Attach n handles, one per generator, to get $\mathbb{M}$
- A word is a loop in the manifold: $\alpha_1^{-1}\alpha_3\alpha_4\alpha_2$, $\alpha_2\alpha_1^{-1}\alpha_3\alpha_4$

RELATIONS



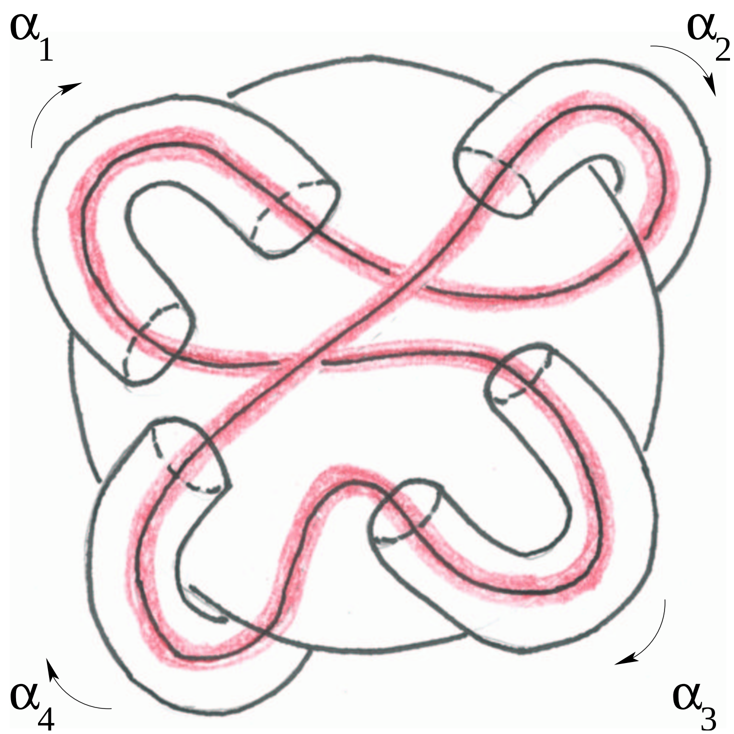
- $r_i = 1$ means loop C_i in $\mathbb{M}$ should be trivial
- Need a disk to make C_i bounding

PRODUCT SPACES



(a) $\mathbb{S}^0 \times \mathbb{B}^2$ (b) $\mathbb{S}^0 \times \mathbb{S}^1$ (c) $\mathbb{B}^1 \times \mathbb{S}^1$

DEHN SURGERY



- Carve out $N_i \approx \mathbb{S}^1 \times \mathbb{B}^3$
- $\partial N_i \approx \mathbb{S}^1 \times \mathbb{S}^2$
- $\partial(\mathbb{B}^2 \times \mathbb{S}^2) \approx \mathbb{S}^1 \times \mathbb{S}^2$, so sew it in

THE REDUCTION

- Use Dehn surgery to kill all r_i and get $\mathbb{M}_m$
- If we have an algorithm to find whether $\mathbb{M}_m$ is homeomorphic to $\mathbb{S}^4$, we have one for determining whether a finitely-presented group is trivial.
- Works for $n \geq 4$
- Works for homotopy $n \geq 4$
- Works for any "interesting property" (Rice Theorem)