Expansive Motions and the Polytope of Pointed Pseudo-Triangulations

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Abstract

We introduce the polytope of pointed pseudo-triangulations, defined as the polytope of expansive motions of a planar point set subject to certain constraints on the increase of their distances. Its 1-skeleton is the graph whose vertices are the pointed pseudo-triangulations of the point set and whose edges are flips of interior pseudo-triangulation edges.

For points in convex position we obtain a new realization of the associahedron, i.e., a geometric representation of the set of triangulations of an n-gon, or of the set of binary trees on n vertices, or of many other combinatorial objects that are counted by the Catalan numbers. By considering the 1-dimensional version of the polytope of constrained expansive motions we obtain a second distinct realization of the associahedron.

Our polytopes have a large number of free parameters, leading to entire families of distinct representations. In particular, our associahedra are distinct from the other previously known realizations.

1 Introduction

Polytopes for combinatorial objects. Describing all instances of a combinatorial structure (e.g. trees or triangulations) as vertices of a polytope is often a step towards giving efficient optimization algorithms on those structures. It also leads to quick prototypes of enumeration algorithms using known vertex enumeration techniques and existing code [2, 7].

One particularly nice example is the associahedron, (see Figure 4 for an example): the vertices of this polytope correspond to Catalan structures. The Catalan structures refer to any of a great number of combinatorial objects which are counted by the Catalan numbers (see the extensive list in Stanley [19, ex. 6.19, p. 219]). Some of the most notable ones are the triangulations of a convex polygon, binary trees, the ways of evaluating a product of n factors when multiplication is not associative (hence the name associahedron), and monotone lattice paths that go from one corner of a square to the opposite corner without crossing the diagonal.

In this paper we describe a new polyhedron whose vertices correspond to *pointed pseudo-triangulations*.

Pseudo-triangulations. Pseudo-triangulations, and the closely related geodesic triangulations of simple polygons, have been used in Computational Geometry in applications such as ray shooting [10], visibility [16], and kinetic data structures [1, 12]. The *minimum* or *pointed pseudo-triangulations* introduced in Streinu [20] have applications to non-colliding motion planning of planar robot arms. For points in convex position, pseudo-triangulations coincide with triangulations, and in this case, our construction gives a new realization of the associahedron.

Expansive motions. An expansive motion is an infinitesimal motion of a set of points where no distance between two vertices decreases. Expansive motions in the plane were instrumental in showing that every simple polygon in the plane can be unfolded into convex position without collisions, see

Connelly, Demaine and Rote [5] and Streinu [20]. These motions form a polyhedral cone (the *expansion cone*) whose extreme rays correspond to equivalence classes of pointed pseudo-triangulations with one convex hull edge removed, modulo rigid subcomponents. A by-product of approach is a new proof for the existence of expansive motions for non-convex polygons and polygonal arcs.

In this paper we introduce *constrained expansions* as expansive motions with a special lower bound on the edge length increase. They form a polyhedron which is a perturbation of the expansion cone. This is the polyhedron mentioned above, whose vertices correspond to the pointed pseudo-triangulations. Adjacency on the polyhedron reflects a certain neighborhood structure among pointed pseudo-triangulations (flips of interior edges). We also investigate 1-dimensional expansive motions, and they give rise to yet another realization of the associahedron.

Rigidity. The connection of these results with rigidity theory is also worth mentioning. Pointed pseudo-triangulations are special instances of minimally rigid frameworks in dimension 2, whose combinatorial structure is well understood. One-dimensional minimally rigid frameworks are trees, another well understood combinatorial structure (see [11]). Adding the constraint of expansiveness is what leads to pointed pseudo-triangulations in 2d, and to the special trees considered in this paper in 1d.

Future perspectives. It is our hope that the insight into one- and two-dimensional motions may eventually lead to generalizations to space and higher dimensions. The 3-dimensional version of the robot arm motion planning problem, with potential applications to computational biology (protein folding), is much more challenging.

Overview. In Section 2 we give the preliminary definitions and results. Section 3 contains the main result, the construction of the polytope of pointed pseudo-triangulations (*ppt-polytope*). Section 4 applies the main result to get a new proof for the existence of expansive motions for non-convex polygons and polygonal arcs in the plane. In Section 5 we present an alternative construction and two special cases leading to the associahedron: points in convex position and the polytope of constrained expansions in dimension 1. Section 6 attempts to put the results in 1 and 2 dimensions into a broader perspective, with the aim of extending the results to higher dimensions and to point sets which are not in general position. We conclude with some final comments in Section 7.

2 Preliminaries

Abbreviations and conventions. Throughout this paper we will assume general position for our point sets, i.e. we assume that no $d+1$ points in $\mathbb{R}^d$ lie in the same hyperplane (unless otherwise specified). We abbreviate "polytope of pointed pseudo-triangulations" as *ppt-polytope*, "one-degree-of-freedom mechanism" as *1DOF mechanism* and "pseudo-triangulation expansive mechanism" as *pte-mechanism.*

Pseudo-triangulations. A *pseudo-triangle* is a simple polygon with only three convex vertices (called *corners*) joined by three inward convex polygonal chains. In particular, a triangle is a pseudo-triangle. A *pseudo-triangulation* is a partitioning of the convex hull of a point set $P = \{p_1, \dots, p_n\}$ into pseudo-triangles using P as vertex set. Pseudo-triangulations are *graphs embedded on* P, i.e. graphs drawn in the plane on the vertex set P and with straight-line segments as edges.

We will work with other graphs embedded in the plane. If edges intersect only at their end-points, as is the case for pseudo-triangulations, the graphs will be called *non-crossing* or *plane graphs.* A graph is *pointed at a vertex* v if there is (locally) an angle at v strictly larger than π and containing no edges. Under our general position assumption, convex-hull vertices are pointed for any plane graph, as are vertices of degree at most two. A graph is called *pointed* if it is pointed at every vertex. Fig. 1 shows a pointed pseudo-triangulation.

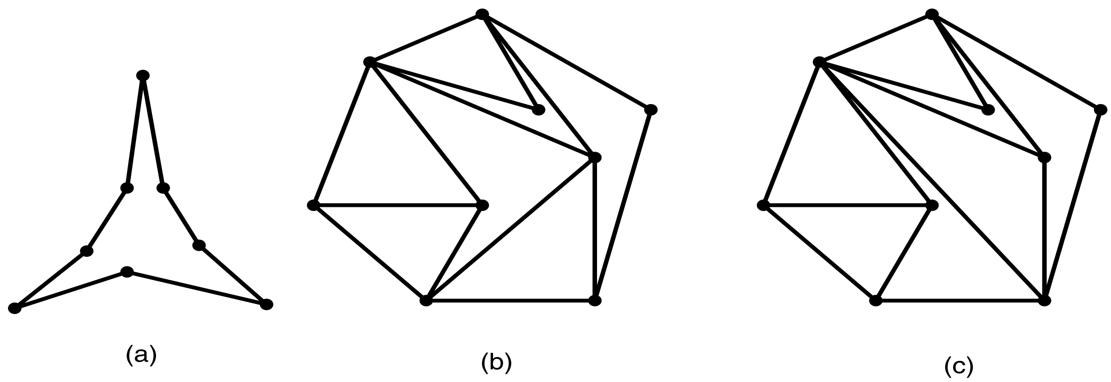


Figure 1: (a) A pseudo-triangle. (b) A minimum, or pointed, pseudo-triangulation. (c) A flip in the pseudo-triangulation from (b).

Lemma 2.1. (Properties of pointed pseudo-triangulations, Streinu [20]) *Let $G = (P, E)$ be a graph on n points in the plane.*

1. **(Characterization)** *The following properties are equivalent.*
   (a) *G is a pseudo-triangulation with the minimum possible number of edges.*
   (b) *G is a pointed pseudo-triangulation.*
   (c) *G is a pseudo-triangulation with exactly $2n-3$ edges (and hence $n-2$ faces).*
   (d) *G is non-crossing, pointed and has $2n-3$ edges.*
   (e) *G is non-crossing, pointed and maximal with this property.*

2. *If any of the above conditions is satisfied, then:*
   (a) *The subgraph induced on any subset set of k vertices has at most $2k-3$ edges (the* hereditary property*). Therefore G is a* **Laman graph** *(see the definition below in the paragraph on rigidity).*
   (b) **(Flips)** *If an interior edge (not on the convex hull) is removed from a pointed pseudo-triangulation, there is a unique way to put back another edge to obtain a different pointed pseudo-triangulation.*
   (c) *(***Connectivity of the flip graph***) The graph whose vertices are pointed pseudo-triangulations and whose edges correspond to flips of interior edges is connected.*

The embedded graphs characterized by the previous lemma will be called either *pointed* or *minimum pseudo-triangulations* (shortly *ppt*). They are the main objects of study in this paper.

Proof. All statements except 2.(c) can be found in [20]. For completeness, we include an easy proof of 2.(c), because it is crucial for our main result (see also [3] for a similar proof). Let p_n be a convex hull vertex. Pointed pseudo-triangulations in which p_n is not incident to any interior edge are just pointed pseudo-triangulations of $P \setminus \{p_n\}$ together with the two "tangent" edges from p_n to the convex hull of the rest. By induction, we assume all those pointed pseudo-triangulations to be connected to each other. To get from an arbitrary pointed pseudo-triangulation to one of them just observe that if a pointed pseudo-triangulation has an interior edge incident to p_n, then a flip on that edge inserts an edge not incident to p_n, hence decreasing the number of interior edges incident to p_n. (This property of flipping does not hold in general, but it certainly holds when p_n is a hull vertex.) □

Rigidity. A graph $G = (P, E)$ embedded on a set of points $P = \{p_1, \dots, p_n\} \in \mathbb{R}^d$ is customarily called a *framework* and denoted by $G(P)$ in rigidity theory. In this paper we work mostly with points in dimensions $d = 2$ and $d = 1$.

An *infinitesimal motion* is an assignment of a velocity vector $v_i = (v_i^1, \dots, v_i^d)$ to each point p_i, $i = 1, \dots, n$. The *trivial infinitesimal motions* are those which come from (infinitesimal) rigid transformations of the whole ambient space. In $\mathbb{R}^2$ these are the translations (for which all the v_i's are equal vectors) and rotations with a certain center p_0 (for which each v_i is perpendicular and proportional to the segment p_0p_i). Trivial motions form a linear subspace of dimension $\binom{d+1}{2}$ in the linear space $(\mathbb{R}^d)^n$ of all infinitesimal motions. Two infinitesimal motions whose difference is a trivial motion will be considered equivalent, leading to a reduced space of *non-trivial* infinitesimal motions of dimension $dn - \binom{d+1}{2}$. In particular, this is $n-1$ for $d = 1$ and $2n-3$ for $d = 2$. Rather than performing a formal quotient of vector spaces we will fix $\binom{d+1}{2}$ variables. E.g. for $d = 1$ we can choose:

$$v_1 = 0 \tag{1}$$

and for $d = 2$ (assuming w.l.o.g. that $p_2^2 \neq p_1^2$):

$$v_1^1 = v_1^2 = v_2^1 = 0 \tag{2}$$

Here, p_1 and p_2 can be any two vertices. A different choice of normalizing conditions only amounts to a linear transformation in the space of infinitesimal motions.

An infinitesimal motion such that $\langle p_i - p_j, v_i - v_j \rangle = 0$ for every edge $ij \in E$ is called a *flex* of $G(P)$. This condition states that the length of the edge ij remains unchanged. The trivial motions are the flexes of the complete graph. A graph (by which we mean an embedded graph or framework) is *infinitesimally rigid* if there are no non-trivial flexes. It is *infinitesimally flexible* or an *infinitesimal mechanism* otherwise. The set of non-trivial infinitesimal motions of a mechanism is a linear subspace, whose dimension (after identifying motions whose difference is trivial) is its *degree of freedom*. The stronger concept of a framework being *rigid* (for which we refer the reader to the rigidity theory literature [11, 21]) is not strictly necessary for the purposes of this paper. We will use it occasionally in these preliminaries. Infinitesimally rigid frameworks are rigid.

A graph G is *generically rigid* if it is rigid in almost all embeddings. It is *minimally rigid* if it is rigid and the removal of any edge invalidates this property. Generical minimally rigid graphs in 1d and 2d are well understood. In 1d they are exactly the trees, and in 2d they have a well known characterization due to Laman [13] (see also [11]): graphs with exactly $2n-3$ edges, such that any subset of k vertices spans at most $2k-3$ edges. We will refer to these graphs as *Laman graphs*.

Finally, *expansive (infinitesimal) motions* $v_1, \dots, v_n$ are those which simultaneously increase (perhaps not strictly) all the pairs of distances: $\langle p_i - p_j, v_i - v_j \rangle \geq 0$ for every $ij \in E$. A mechanism is *expansive* if it has non-trivial expansive flexes.

Proposition 2.2. (Rigidity properties of pointed pseudo-triangulations, Streinu [20])

(a) *Pseudo-triangulations are minimally infinitesimally rigid (and therefore rigid).*

(b) *The removal of a convex hull edge from a pointed pseudo-triangulation yields a 1DOF expansive mechanism (called a* pseudo-triangulation expansive mechanism *or shortly a* pte-mechanism*).* □

Part (a) is in accordance with the fact that pseudo-triangulations are Laman graphs, and therefore generically rigid. It is a trivial consequence of (a) that the removal of an edge creates a (not necessarily expansive) 1DOF mechanism. The expansiveness of pte-mechanisms (part (b)) was proved in [20] using the Maxwell-Cremona correspondence between self-stresses and 3-d liftings of planar frameworks, a technique that was introduced in [5]. Below we will give an independent proof of this proposition as a consequence of Theorem 3.1.

The *(infinitesimal) rigidity map* $M_{G(P)} : (\mathbb{R}^d)^n \to \mathbb{R}^{E(G)}$ is a linear map associated to an embedded framework $G(P)$, $P \subset \mathbb{R}^d$. It sends each infinitesimal motion $(v_1, \dots, v_n) \in (\mathbb{R}^d)^n$ to the vector of infinitesimal edge increases $(\langle p_i - p_j, v_i - v_j\rangle)_{ij\in E} \in \operatorname{Im} M$. When no confusion arises, it will be simply denoted as M. As usual, the image of M is denoted by $\operatorname{Im} M = \{ f \mid f = Mv \}$. In the matrix of M the row indexed by the edge $ij \in E$ has 0 entries everywhere, except in the i-th and j-th group of d columns, where the entries are $p_i - p_j$, resp. $p_j - p_i$. The kernel of M is the space of flexes of $G(P)$. In particular, a graph (framework) is rigid if and only if the kernel of its associated rigidity map M is the subspace of trivial motions.

Self-stresses. A *self-stress* (or an *equilibrium stress*) on a framework $G(P)$ (see [21] or [5, Section 3.1]) is an assignment of scalars ω_{ij} to edges such that $\forall i \in P$, $\sum_{ij\in E} \omega_{ij}(p_i - p_j) = 0$. The proof of the following lemma is straightforward.

Lemma 2.3. *Self-stresses form the orthogonal complement of the linear subspace* $\operatorname{Im} M \subset \mathbb{R}^{\binom{d}{2}}$. *In other words,* $(\omega_{ij})_{ij\in E}$ *is a self-stress if and only if for every infinitesimal motion* $(v_1, \ldots, v_n) \in (\mathbb{R}^d)^n$ *the following identity holds:*

$$\sum_{ij\in E} \omega_{ij}\langle p_i - p_j, v_i - v_j\rangle = 0 \qquad \square$$

The special case $d = 2$, $n = 4$ will be relevant to our purposes, so we analyze it separately. Let p_1, p_2, p_3 and p_4 be four points in $\mathbb{R}^2$, considered as an embedding of the complete graph on 4 points. $\operatorname{Im} M$ has at most dimension $2n-3 = 5$ and hence the space of self-stresses must have at least dimension 1. It turns out that under our non-collinearity assumption, it always has dimension 1. We can actually describe explicitly the unique (up to a constant factor) self-stress. For any three points $p, q, r \in \mathbb{R}^2$ we denote by $\det(p, q, r)$ the determinant of the three vectors $(p, 1)$, $(q, 1)$, $(r, 1) \in \mathbb{R}^3$, in this order. Equivalently, $\det(p, q, r)$ equals twice plus or minus the area of the triangle (p, q, r), with positive sign iff p, q and r are in counterclockwise order.

Lemma 2.4. *The following gives a self-stresses for any four points in general position in the plane:*

$$\omega_{ij} := \frac{1}{\det(p_i, p_j, p_k)\det(p_i, p_j, p_l)} \tag{3}$$

where k and l are the two indices other than i and j.

Proof. For symmetry reasons, it suffices to verify the definition of self-stress only at p_1:

$$\omega_{12}(p_1 - p_2) + \omega_{13}(p_1 - p_3) + \omega_{14}(p_1 - p_4) = 0$$

After plugging in the definition (3) of ω_{ij} and discarding the denominator, the left-hand side becomes

$$\det(p_1, p_3, p_4)(p_1 - p_2) - \det(p_1, p_2, p_4)(p_1 - p_3) + \det(p_1, p_2, p_3)(p_1 - p_4) \tag{4}$$

If $p_i = (x_i, y_i)$, then the x-coordinate of (4) is the determinant of the matrix whose columns are the four vectors $(x_j, y_j, 1, x_j - x_1)$, $j = 1, \ldots, 4$, expanded along the last row. This determinant is zero because the fourth row of the matrix is a linear combination of the first and third rows. The y-coordinate of (4) is the determinant of the 4×4 matrix with rows $(x_j, y_j, 1, y_j - y_1)$, which is zero by a similar argument. $\square$

The self-stress of four points is unique up to a constant factor. Multiplying the formulas (3) by the product of the four determinants of the four points we would get somewhat simpler formulas, each ω_{ij} becoming the product of two determinants, instead of the inverse of such a product. Our choice is more convenient because no matter which four points we take, the ω_{ij}'s will be positive for convex hull edges and negative for interior edges. Note that a graph with five edges on four points is non-crossing and pointed (i.e. a pointed pseudo-triangulation) if and only if the missing edge is interior. We will use these observations in our construction of the ppt-polytope.

The expansion cone. We are given a set of n points $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ that are to move with (unknown) velocities $v_i \in \mathbb{R}^d$, $i = 1, \ldots, n$. An *expansive* motion is a motion in which no inter-point distance decreases. This is described by the system of homogeneous linear inequalities:

$$\langle p_i - p_j, v_j - v_i \rangle \geq 0, \qquad \forall\ 1 \leq i < j \leq n \tag{5}$$

and hence defines a polyhedral cone. Since the only motions in the intersection of all facets of the cone are the trivial motions, in the reduced space of infinitesimal motions (with normalizing equations like (1) or (2)) we get a *pointed polyhedral cone* containing the origin as a vertex. We call it the cone of expansive motions or simply the *expansion cone* $\bar{X}_0(P)$. (The reason for this notation will become clear later.)

An extreme ray of the expansion cone is given by a maximal set of inequalities satisfied with equality by non-trivial motions. Each inequality corresponds to an edge of the point set, so that the ray corresponds to a graph embedded in our point set. The cardinality of this set of edges is at least the dimension of the cone minus 1, but may be much larger. Let's analyze the low dimensional cases.

For $d = 1$ the expansion cone is not very interesting. Let's assume that the points $p_i \in \mathbb{R}$ are labeled in increasing order $p_1 < p_2 < \cdots < p_n$. Then:

Proposition 2.5. *The expansion cone in one dimension has $n-1$ extreme rays corresponding to the motions where $p_1, \ldots, p_i$ remain stationary and the points $p_{i+1}, \ldots, p_n$ move away from them at uniform speed:*

$$0 = v_1 = v_2 = \cdots = v_i < v_{i+1} = \cdots = v_n \tag{6}$$

Proof. Note that the actual values of p_i are immaterial in this case. The expansion cone is given by the linear system $v_j \geq v_i$, $1 \leq i \leq j \leq n$ plus the extra condition $v_1 = 0$, and any maximal set of inequalities satisfied with equality and yet not trivial is obviously given by (6). □

The 2d case is more complex and requires additional terminology. Generic 1DOF mechanisms contain rigid subcomponents (r-components, cf. [11]): maximal sets of some k vertices spanning a Laman subgraph on $2k - 3$ edges. The r-components of pte-mechanisms are themselves pseudo-triangulations spanning convex subpolygons including all points in their interior. Adding edges to complete each r-component to a complete subgraph yields a *collapsed pte-mechanism* (see Figures 2 and 3).

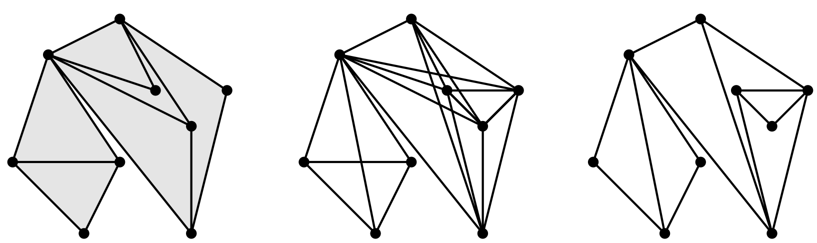


Figure 2: A pte-mechanism with rigid sub-components (convex subpolygons) drawn shaded, the corresponding collapsed pte-mechanism, and another pte-mechanism that yields the same expansive motion.

Proposition 2.6. *In dimension 2, the extreme rays of the expansion cone $\bar{X}_0$ correspond to the collapsed pte-mechanisms.* □

The proof will be given in Section 4.1, after we have determined the extreme rays of a perturbed version of the polytope.

The polytope of constrained expansions. The construction we will give in section 3 can roughly be interpreted as separating the pseudo-triangulations contained in the same collapsed pte-mechanisms, to obtain a polyhedron whose vertices correspond to distinct pseudo-triangulations. The original expansion cone is highly degenerate: its extreme rays contain information about *all* the bars whose length is

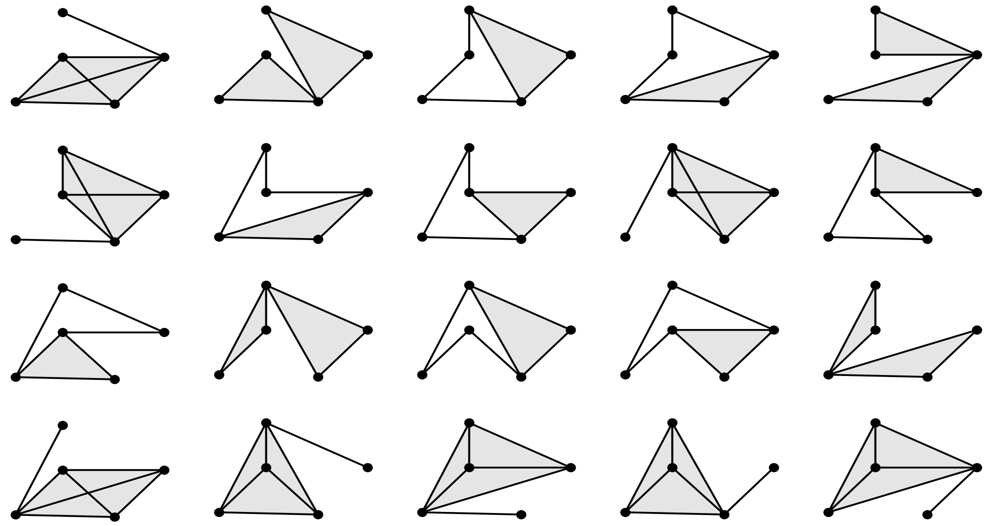


Figure 3: The collapsed pte-mechanisms corresponding to the 20 extreme rays of the expansion cone $\bar{X}_0$ for a planar point set of 5 points. The rigid sub-components (complete subgraphs) are shaded.

unchanged by a motion of one 1DOF expansive mechanism. We would like to perturb the constraints (5) to eliminate these degeneracies and recover pure pseudo-triangulations. We do so by giving up homogeneity, i.e., by translating the facets of the expansion cone. Our system will become:

$$\langle v_j - v_i, p_i - p_j \rangle \geq f_{ij}, \qquad \forall 1 \leq i < j \leq n \tag{7}$$

for some numbers f_{ij}. In some cases we will change these inequalities to equations for the edges on the convex hull of the given point set.

$$\langle v_j - v_i, p_i - p_j \rangle = f_{ij}, \qquad \text{for the convex hull edges } ij. \tag{8}$$

Section 3 proves our main result, Theorem 3.1: For any point set in general position in the plane and for some appropriate choices of the parameters f_{ij}, (7) defines a polyhedron whose vertices are in bijection with pointed pseudo-triangulations and all lie in a unique maximal bounded face given by (8): the "polytope of pointed pseudo-triangulations". A similar thing in 1d is done in Section 5.3, with the surprising outcome that the (unique) maximal bounded face of the polyhedron turns out to be an associahedron with vertices corresponding to *non-crossing alternating trees* (which are Catalan structures, as shown in [8]). The next paragraph prepares the ground for this result.

The associahedron. The associahedron is a polytope which has a vertex for every triangulation of a convex n-gon, and in which two vertices are connected by an edge of the polytope if the two triangulations are connected by an edge flip. Equivalently, various types of Catalan structures are reflected in the associahedron. Fig. 4 shows an example.

There is an easy geometric realization of this polytope associated to each set P of n points *in convex position* in the plane, as a special case of a secondary polytope (Gel′fand, Zelevinskiĭ, and Kapranov [9], see also Ziegler [22, Section 9.2]). Every triangulation is represented by a vector $(a_1, \ldots, a_n)$ of n components. The entry a_i is simply the sum of the areas of all triangles of the triangulation that are incident to the i-th vertex. We will refer to this realization as the *classical realization* of the associahedron. It depends on the location of the vertices of the convex n-gon, but all polytopes that one gets in this way are combinatorially equivalent. Their face lattice is the poset of polygonal subdivisions of the n-gon or, in the terminology of the previous paragraphs, non-crossing and pointed graphs embedded in P and containing the n convex hull edges. But observe that the word "pointed" is superfluous for a graph with vertices in convex position. The order structure in this poset is just inclusion of edge sets (in reverse order since maximal graphs represent vertices).

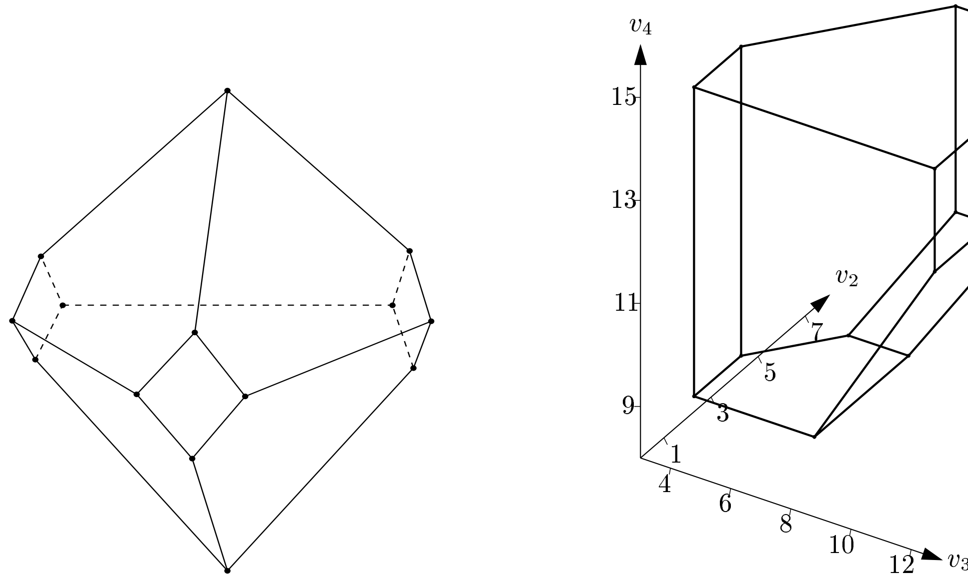


Figure 4: The three-dimensional associahedron. The vertices represent all triangulations of a convex hexagon or all possible ways to insert parentheses into the product $a * b * c * d * e$. Left: a symmetric representation. Right: our representation, from Section 5.3.

Dantzig, Hoffman, and Hu [6, Section 2], and independently de Loera et al. [15] in a more general setting, have given other representations of the set of triangulations as the vertices of a 0-1-polytope in $\binom{n}{3}$ variables corresponding to the possible triangles of a triangulation (the *universal* polytope), or in $\binom{n}{2}$ variables corresponding to the possible edges of a triangulation. These realizations are in a sense most natural, but they have higher dimensions and have more adjacencies between vertices than the associahedron. Every classical associahedron, however, arises as a projection of the universal polytope. The first published realization of an associahedron is due to Lee [14], but it is not fully explicit. A few earlier and more complicated ad-hoc realizations that were never published are mentioned in Ziegler [22, Section 0.10].

In this paper the associahedron appears in two forms. First, we will show that for n points in convex position our polytope of pointed pseudo-triangulations is affinely equivalent to the secondary polytope of the configuration, which is a classical associahedron (Section 5.2). Second, as mentioned before, our construction adapted to a one-dimensional point configuration produces in a natural way an associahedron (Section 5.3).

Notice that in dimension 1 the coordinates p_i can be eliminated from the constraints (7). Only the order of points along the line matters. One can also look at the whole *arrangement* of hyperplanes of the form

$$v_j - v_i = g_{ij}.$$

This arrangement, for various special values of g, has been the object of extensive combinatorial studies. For $g \equiv 0$ it is the classical *Coxeter* or *reflection* arrangement of type A_n. The case $g \equiv 1$ has been studied by Postnikov and Stanley [18]. The expansion cone of a 1d point set is the *positive cell* in the arrangement A_n, and our associahedron is a bounded face of the polyhedron obtained by translating the facets of this cell. It is interesting that these new associahedra are not affinely equivalent to any classical associahedron obtained as a secondary polytope. Also, that we are trying to get a simple polyhedron, in contrast to the above-mentioned choices of g which lead to highly degenerate arrangements.

3 The Main Result: the Polytope of Pointed Pseudo-Triangulations

In this section we prove our main result.

Theorem 3.1. *For every set $P = \{p_1, \ldots, p_n\}$ of $n \geq 3$ planar points in general position, there is a choice of f_{ij}'s for which equations* (7) *together with the normalizing equations* (2) *define a simple polyhedron of dimension $2n-3$ with the following properties:*

1. *The face poset of the polyhedron equals the opposite of the poset of pointed and non-crossing graphs on P, by the map sending each face to the set of edges whose corresponding equations* (7) *are satisfied with equality over that face. In particular:*
   - *(a) Vertices of the polyhedron are in 1-to-1 correspondence with pointed pseudo-triangulations of P.*
   - *(b) Bounded edges correspond to flips of interior edges in pseudo-triangulations, i.e., to pseudo-triangulations with one interior edge removed.*
   - *(c) Extreme rays correspond to pseudo-triangulations with one convex hull edge removed.*
2. *The face obtained by changing to equalities* (8) *those inequalities from* (7) *which correspond to convex hull edges of P is bounded (hence a polytope) and contains all vertices. In other words, it is the unique maximal bounded face, and its 1-skeleton is the graph of flips among pointed pseudo-triangulations.*

The proof is a consequence of lemmas proved throughout this section. Actually, Theorem 3.9 states that either of the two choices $f_{ij} := \det(0, p_i, p_j)^2$ or $f_{ij} := \left(|p_i|^2 + |p_j|^2 + \langle p_i, p_j\rangle\right) |p_i - p_j|^2$ produce the desired object. In Section 5.1 we will study the space of all "valid choices" and derive a more canonical description of the polyhedron (and the polytope) in question.

Before going on, let us see that Theorem 3.1 implies Proposition 2.2. Observe that a framework is minimally infinitesimally rigid if and only if the hyperplanes $\langle p_i - p_j, v_i - v_j\rangle = 0$ corresponding to its edges ij meet transversally and at a single point, in the $(2n-3)$-dimensional space given by equations (2). Our theorem says that this happens for the $2n-3$ translated hyperplanes $\langle p_i - p_j, v_i - v_j\rangle = f_{ij}$ corresponding to any pointed pseudo-triangulation, hence giving part (a) of Proposition 2.2. An (infinitesimally) expansive 1DOF mechanism is one whose corresponding hyperplanes intersect in a line contained in the expansion cone, which is just the recession cone of our polyhedron. Hence, Part 1c of Theorem 3.1 implies part (b) of Proposition 2.2.

The polyhedron and the polytope of constrained expansions. The solution set $v \in \left(\mathbb{R}^2\right)^n$ of the system of inequalities (7) together with the normalizing equations (2) will be called the *polyhedron of constrained expansions* $\bar{X}_f(P)$ for the set of points P and perturbation parameters (constraints) f. We will frequently omit the point set P when it is clear from the context. A solution v may satisfy some of the inequalities in (7) with equality: the corresponding edges $E(v)$ of G are said to be *tight* for that solution. In the same way, for a face K of $\bar{X}_f$ we call *tight edges* of K and denote $E(K)$ the edges whose equations are satisfied with equality over K (equivalently, over a relative interior point of K). This is the correspondence that Theorem 3.1 refers to: the edges $E(K)$ of a face K form the pointed and non-crossing graph corresponding to that face.

When $f \equiv 0$, we just get the expansion cone $\bar{X}_0$ itself, which is the recession cone of $\bar{X}_f$, for any choice of f. (In this sense, our notations $\bar{X}_0$ and $\bar{X}_f$ are consistent.) We will first establish a few properties of the expansion cone.

Lemma 3.2. (a) *The expansion cone $\bar{X}_0$ is a pointed polyhedral cone of full dimension $2n-3$ in the subspace defined by the three equations* (2).

(b) *Consider the set $E(v)$ of tight edges for any feasible point $v \in \bar{X}_0$. If $E(v)$ contains*

(i) two crossing edges,

(ii) a set of edges incident to a common vertex with no angle larger than π (witnessing that $E(v)$ is not pointed at this vertex), or

(iii) a convex subpolygon,

then $E(v)$ must contain the complete graph between the endpoints of all involved edges. In case (iii), this complete graph also includes all points inside the convex subpolygon.

Proof. (a) The dilation (scaling motion) $v_i := p_i$ satisfies all inequalities (5) strictly. By adding a suitable rigid motion, the three equations (2) can be satisfied, too, without changing the status of the inequalities (5), and so we get a relative interior point in the $(2n-3)$-dimensional subspace (2).

If the cone were not pointed, it would contain two opposite points v and $-v$. From this we would conclude that $\langle v_j - v_i, p_j - p_i \rangle = 0$ for all i, j, and hence v would be a flex of the complete graph on P, i.e., a trivial flex. By the normalizing equations (2), v must then be 0.

(b) We first consider (iii), which is the most involved case. Let v be an expansive motion which preserves all edge lengths of some convex polygon. First we see that v preserves all distances between polygon vertices: indeed, if it preserves lengths of polygon edges but is not a trivial motion of the polygon then the angle at some polygon vertex p_i infinitesimally decreases, because the sum of angles remains constant. But decreasing the angle at p_i while preserving the lengths of the two incident edges implies that the distance between the two vertices adjacent to p_i in the polygon decreases. This is a contradiction.

By choosing p_1 and p_2 to be polygon vertices, the above implies that the polygon remains stationary under v. Now no interior point p_i can move with respect to the polygon, without decreasing the distance to some polygon vertex: If $v_i \neq 0$, there is at least one hull vertex p_j in the half-plane $\langle p_i - p_j, v_i \rangle < 0$. The edge ij will then violate condition (5).

Case (ii) is similar: If the edges incident to a vertex p_i do not move rigidly, at least one angle between two neighboring edges must decrease, and, this angle being less than π, this implies that the distance between the endpoints of these edges decreases, a contradiction.

For case (i), we apply Lemma 2.3 to our given four-point set in convex position, with the self-stress of Lemma 2.4, which is positive for the four hull edges and negative for the two diagonals. This implies that this four-point set can have no non-trivial expansive motion which is not strictly expansive on at least one of the two diagonals. □

As an immediate consequence of Lemma 3.2(a), we get:

Corollary 3.3. *$\bar{X}_f(P)$ is a $(2n-3)$-dimensional unbounded polyhedron with at least one vertex, for any choice of parameters f.* □

It is easy to derive part 2 of Theorem 3.1 from part 1. For every vertex or bounded edge of $\bar{X}_f(P)$, the set $E(v)$ contains all convex hull edges of P. On the contrary, for any unbounded edge (ray) of $\bar{X}_f(P)$, the set $E(v)$ misses some convex hull edge of P. Hence, by setting to equalities the inequalities corresponding to convex hull edges we get a face $X_f(P)$ of $\bar{X}_f(P)$ which contains all vertices and bounded edges of $\bar{X}_f(P)$, but no unbounded edge.

In order to prove part 1, we first need to check that indeed X_f is a bounded face, and hence a polytope which we call the *polytope of constrained expansions* or pce-polytope for the set of points P and perturbation parameters f.

Lemma 3.4. *For any choice of f, $X_f(P)$ is a bounded set.*

Proof. Suppose that $v_0 + tv$ is in X_f for all $t \geq 0$. Then we must have $v \in X_0$. Hence, it suffices to show that $X_0 = 0$, i.e. that the graph consisting of all convex hull edges has no non-trivial expansive flexes. This is an immediate consequence of Lemma 3.2b(iii). □

Valid choices of the perturbation parameters. We call a choice of the constants $f = (f_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ *valid* if the corresponding polyhedron $\bar{X}_f$ of constrained expansions has the combinatorial structure claimed in Theorem 3.1.

Lemma 3.5. *A choice of $f \in \mathbb{R}^{\binom{n}{2}}$ is valid if and only if the graph $E(v)$ of tight edges corresponding to any feasible point $v \in \bar{X}_f(P)$ is non-crossing and pointed.*

Proof. Necessity is trivial, by definition of being valid. To see sufficiency note that, by Corollary 3.3, $\bar{X}_f$ has dimension $2n-3$. Thus, any vertex v of the polyhedron is incident to at least $2n-3$ faces $E(v)$. If $E(v)$ is non-crossing and pointed, Lemma 2.1(1) implies that $E(v)$ has *exactly* $2n-3$ incident faces and is a pointed pseudo-triangulation. In particular, the polyhedron is simple. Also, since the tight edges of faces incident to v are different subgraphs of $E(v)$, the poset of faces incident to the vertex v is the poset of all subgraphs of the pointed pseudo-triangulation $E(v)$.

It remains only to show that every pointed pseudo-triangulation actually appears as a vertex, for which we use a somewhat indirect argument, based on the fact that the flip graph is connected. This type of argument has also been used by Carl Lee [14] for the case of the associahedron, where it is attributed to Gil Kalai and Micha Perles.

Since the polytope is simple, every vertex v is incident to $2n-3$ edges of $\bar{X}_f$. The sets of tight edges corresponding to them are the $2n-3$ subgraphs of $E(v)$ obtained removing a single edge. We denote K_{ij} the one polyhedral edge corresponding to the removal of ij. By Lemma 3.4, if ij is interior then K_{ij} is bounded. Hence, it is incident to another vertex, which must correspond to a pointed pseudo-triangulation that completes $E(v) - \{ij\}$. By Lemma 2.1(2b), this can only be the one obtained from $E(v)$ by a flip at ij. Together with the fact that the flip graph is connected (Lemma 2.1(2c)) and that $\bar{X}_f$ has at least one vertex, this implies that all pointed pseudo-triangulations appear as vertices of $\bar{X}_f$, and hence that all pointed and non-crossing graphs appear as well.

Also, the extreme rays have the structure predicted in Theorem 3.1. For a convex hull edge ij, K_{ij} must be an unbounded edge because there is no other pointed pseudo-triangulation that contains $E(v) - \{ij\}$. □

Reducing the problem to four points. We now conclude that valid perturbation vectors $f \in \mathbb{R}^{\binom{n}{2}}$ can be recognized by looking at 4-point subsets only.

Lemma 3.6. *A choice of $f \in \mathbb{R}^{\binom{n}{2}}$ is valid if and only if it is valid when restricted to every four points of P.*

Proof. By the previous Lemma, if f is not valid for P then there is a point v of $\bar{X}_f$ for which the graph $E(v)$ is either non-pointed or crossing. In either case, there is a subset of four points $P' \subseteq P$ on which the induced subgraph is non-pointed or crossing. Let v' and f' denote v and f restricted to P'. Then, v' is in $\bar{X}_{f'}(P')$ and the graph $E(v')$ is crossing or not pointed, hence f' is not valid on P'. Contradiction. □

The case of four points.

Theorem 3.7. *A choice of perturbation parameters $f \in \mathbb{R}^{\binom{4}{2}}$ on four points $P = (p_1, p_2, p_3, p_4)$ forms a valid choice if and only if*

$$\sum_{1 \le i < j \le 4} \omega_{ij} f_{ij} > 0, \tag{9}$$

where the ω_{ij}'s are the unique self-stress on the four points, with signs chosen as in Lemma 2.4.

For a set of four points $P = (p_1, p_2, p_3, p_4)$, we denote G_{ij} the graph on P whose only missing edge is ij. Recall that the choice of self-stress on four points has the property that G_{ij} is pointed and non-crossing (equivalently, ij is interior) if and only if ω_{ij} is negative.

Since $\bar{X}_f(P)$ is five-dimensional, for every vertex v the set $E(v)$ contains at least five edges. Therefore $E(v)$ is either the complete graph or one of the graphs G_{ij}. Theorem 3.7 is then a consequence of Lemma 3.5 and the following statement.

Lemma 3.8. *Let $R := \sum_{1\le i<j\le 4} \omega_{ij} f_{ij}$. For every edge kl, the following properties are equivalent:*

1. *The graph G_{kl} appears as a vertex of $X_f(P)$.*
2. *R and ω_{kl} have opposite signs.*

Proof. The graph G_{kl} appears as a face if and only if the (unique, since G_{kl} is rigid) motion with edge length increase f_{ij} for every edge ij other than kl has edge length increase on kl greater than f_{ij}. In this case, by rigidity of G_{kl}, the face is actually a vertex. But, for this motion:

$$\begin{aligned} 0 &= \sum_{1\le i<j\le 4} \omega_{ij}\langle p_j - p_i, v_j - v_i\rangle = \sum_{1\le i<j\le 4} \omega_{ij} f_{ij} + \omega_{kl}(\langle p_k - p_l, v_k - v_l\rangle - f_{kl}) \\ &= R + \omega_{kl}(\langle p_k - p_l, v_k - v_l\rangle - f_{kl}). \end{aligned}$$

Hence, $\langle p_k - p_l, v_k - v_l\rangle > f_{kl}$ is equivalent to R and ω_{kl} having opposite sign. □

Observe that the previous lemma implicitly includes the statement that the complete graph appears as a vertex if and only if $R = 0$. The *only if* part of this is actually an easy consequence of Lemma 2.3. In this case X_f degenerates to a single point.

Some valid choices. To complete the proof of Theorem 3.1 we still need to show that valid choices of perturbation parameters exist. We know of two relatively simple ones:

Theorem 3.9. *For any point set $P = \{p_1, \dots, p_n\}$ in general position in the plane, the following two choices of parameters f and f' are valid:*

$$f_{ij} = \left(|p_i|^2 + |p_j|^2 + \langle p_i, p_j\rangle\right) \cdot |p_i - p_j|^2 \tag{10}$$

$$f'_{ij} = \det(0, p_i, p_j)^2 \tag{11}$$

Proof. By Lemma 3.6 it suffices to consider $n = 4$. Let p_1, p_2, p_3 and p_4 be any four points in the plane in general position. Let ω_{ij} be the values given by (3) from Lemma 2.4. Let $R := \sum_{1\le i<j\le 4} \omega_{ij} f_{ij}$ and $S := \sum_{1\le i<j\le 4} \omega_{ij} f'_{ij}$. By Theorem 3.7, we only need to show that $R > 0$ and $S > 0$.

The following three lemmas prove the theorem, by showing respectively that:

- $R = 2S$, for any point set.
- $S = 1$ if one of the points coincides with the origin of coordinates.
- R is invariant under translation of the coordinate system.

□

Lemma 3.10. *$R = 2S$, for any point set.*

Proof. Apply Lemma 2.3 with $v_i = |p_i|^2 p_i$. We get $\sum_{i,j} \omega_{ij}\langle p_j - p_i, |p_j|^2 p_j - |p_i|^2 p_i\rangle = 0$. Expanding the scalar product yields $\sum_{i,j} \omega_{ij} g_{ij} = 0$ with

$$g_{ij} = |p_i|^2|p_i|^2 + |p_j|^2|p_j|^2 - (|p_i|^2 + |p_j|^2)\langle p_i, p_j\rangle.$$

We can rewrite f_{ij} and f'_{ij} as

$$\begin{aligned} f_{ij} &= \left(|p_i|^2 + |p_j|^2\right)^2 - \left(|p_i|^2 + |p_j|^2\right)\langle p_i, p_j\rangle - 2\langle p_i, p_j\rangle^2, \\ f'_{ij} &= |p_i|^2|p_j|^2 - \langle p_i, p_j\rangle^2. \end{aligned}$$

From this, $f_{ij} - 2f'_{ij} = g_{ij}$. Hence, $R - 2S = \sum_{i,j} \omega_{ij} g_{ij} = 0$. □

Lemma 3.11. *If one of the points equals* $(0,0)$*, then* $S = 1$.

Proof. Assume $p_1 = (0,0)$. Then, $f'_{1i} = 0$ for every i and $f'_{ij} = \det(0,p_i,p_j)^2 = \det(p_1,p_i,p_j)^2$ for $i \neq 1 \neq j$. Hence:

$$\begin{aligned} S &= \frac{\det(p_1,p_2,p_3)^2}{\det(p_2,p_3,p_1)\det(p_2,p_3,p_4)} + \frac{\det(p_1,p_2,p_4)^2}{\det(p_2,p_4,p_1)\det(p_2,p_4,p_3)} + \frac{\det(p_1,p_3,p_4)^2}{\det(p_3,p_4,p_1)\det(p_3,p_4,p_2)} = \\ &= \frac{\det(p_1,p_2,p_3)}{\det(p_2,p_3,p_4)} + \frac{\det(p_1,p_2,p_4)}{\det(p_2,p_4,p_3)} + \frac{\det(p_1,p_3,p_4)}{\det(p_3,p_4,p_2)} = \\ &= \frac{1}{\det(p_2,p_3,p_4)}(\det(p_1,p_2,p_3) - \det(p_1,p_2,p_4) + \det(p_1,p_3,p_4)) = 1, \end{aligned}$$

where the last equality follows from $\det(p_2,p_3,p_4) = \det(p_1,p_2,p_3) + \det(p_1,p_3,p_4) + \det(p_1,p_4,p_2)$. □

Let $v \in \mathbb{R}^2$ be a translation vector. Denote $p_i(v) = p_i + v$ and let $f_{ij}(v)$ and $R(v)$ be the quantities f_{ij} and R computed for the translated points. The quantities ω_{ij} and $p_i - p_j$ are invariant under translation.

Lemma 3.12. $R(v) = R(0,0)$ *for any vector* $v \in \mathbb{R}^2$ *(i.e., R is invariant under translation).*

Proof. Let $e_{ij} = p_i - p_j$. Then:

$$\begin{aligned} R(v) &= \sum_{i,j} \omega_{ij}|e_{ij}|^2(|p_i+v|^2 + |p_j+v|^2 + \langle p_i+v, p_j+v\rangle) \\ &= \sum_{i,j} \omega_{ij}|e_{ij}|^2(|p_i|^2 + |p_j|^2 + \langle p_i, p_j\rangle + 3\langle v, p_i+p_j\rangle + 3|v|^2) \\ &= R(0,0) + 3\langle v, \sum_{i,j}\omega_{ij}|e_{ij}|^2(p_i+p_j)\rangle + 3|v|^2\sum_{i,j}\omega_{ij}|e_{ij}|^2. \end{aligned}$$

In the last sum, the third term is zero by Lemma 2.3 taking $v_i = p_i$ (and, hence, $v_j - v_i = e_{ij}$). We conclude that $R(v) = R(0,0) + \langle v, u\rangle$, where u is a certain vector not depending on v. Lemmas 3.10 and 3.11 say that $R(-p_i) = 2$ for every $i \in \{1,2,3,4\}$ and hence $\langle u, e_{ij}\rangle = 0$ for every $i,j \in \{1,2,3,4\}$. This implies $u = (0,0)$. □

This concludes the proof of Theorems 3.9 and 3.1.

4 Applications of the Main Result

4.1 The Expansion Cone

The expansion cone is the recession cone $\bar{X}_0$ of the pce-polyhedron $\bar{X}_f$, whose structure we know. The extreme rays of $\bar{X}_0$ are precisely the extreme rays of $\bar{X}_f$, shifted to start at 0, but parallel rays of $\bar{X}_f$ give of course rise to only one ray of $\bar{X}_0$.

Studying when this happens will allow us to give now a rather easy proof of the characterization of the extreme rays of $\bar{X}_0$ (Proposition 2.6): We conclude from Theorem 3.1 that the extreme rays correspond to pointed pseudo-triangulations with one hull edge removed, i.e., pte-mechanisms. Any convex subpolygon in a pte-mechanism must be rigid in the mechanism, according to Lemma 3.2b(iii). This corresponds to the fact that every convex subpolygon of a pointed pseudo-triangulation contains a pointed pseudo-triangulation of that polygon and the enclosed points, and is therefore rigid. We still have to show that these r-components are the only subcomponents that move rigidly in the (unique) motion v on a pte-mechanism $G(P)$.

Lemma 4.1. *Let* $P' \subset P$ *be a maximal subset that moves rigidly by the unique motion* v *of the pte-mechanism* $G(P)$.

(a) *Then P' contains all points of P within its convex hull,*

(b) *G contains no edge which crosses the boundary of the convex hull of P', and*

(c) *G contains all boundary edges of the convex hull of P'.*

Proof. (a) A subset $P' \subset P$ moves rigidly if and only if $E(v)$, considered in $\bar{X}_0$, contains the complete subgraph spanned by P'. Then part (a) follows from Lemma 3.2b(iii).

(b) An edge $ij \in G \subseteq E(v)$ which crosses a boundary edge $kl \in E(v)$ of the convex hull of P', would, by Lemma 3.2b(i), imply that the complete graph on $\{i, j, k, l\}$ is part of $E(v)$, and hence i and j are rigidly connected to kl and thereby to the whole rigid set P'. Then P' could not be maximal.

(c) Assume that a hull edge ij of P' is missing in G. We know that ij is not the convex hull edge whose insertion produces a pointed pseudo-triangulation, because if that edge were contained in $E(v)$ (together with the remaining hull edges) the whole set P would be rigid. It follows now from part (b) that the addition of ij does not create a crossing or a non-pointed vertex in $E(v)$ and hence in G. This contradicts the maximality property of G as a pte-mechanism (Lemma 2.1(1e)). □

If follows from the last statement that the rigidly moving components are precisely the convex subpolygons of the pte-mechanism, and two pte-mechanisms yield the same motion (extreme ray) if and only if they lead to the same collapsed pte-mechanism, thus concluding the proof of Proposition 2.6. □

4.2 Strictly Expansive Motions and Unfoldings of Polygons

Lemma 4.2. *Let $G(P)$ be a non-crossing and pointed framework in the plane. Then, $G(P)$ has a non-trivial expansive flex if and only if it does not contain all the convex hull edges. In this case, there is an expansive motion that is strictly expansive on all the convex hull edges not in G.*

Proof. If all the convex hull edges are in G, then Lemma 3.4 implies the statement: the face of $\bar{X}_f$ corresponding to G is bounded and, hence, it degenerates to the origin in $\bar{X}_0$. If a certain convex hull edge ij is not in G, then we extend G to a pointed pseudo-triangulation, according to Lemma 2.1(1e). Removing ij yields a pte-mechanism, whose expansive motion is strictly expansive on ij. Adding all such motions for the different missing hull edges gives the stated motion. □

This immediately gives the following theorem.

Theorem 4.3. *Let $G(P)$ be a non-crossing nonconvex plane polygon or a plane polygonal arc that does not lie on a straight line. Then there is a motion that is strictly expansive on at least one edge.* □

This statement has been crucial to show that every simple polygon in the plane can be unfolded into convex position and every polygonal arc can be straightened, without collisions [5, 20]. The proof given in those papers is based on several reduction steps between infinitesimal motions, self-stresses, and polyhedral terrains. The above new proof is completely independent, although not less indirect.

Actually, one can work a little harder in the proof of Lemma 4.2 and show that *any* edge $ij \notin G$ that is not contained inside a convex subpolygon of G can be chosen to be strictly expansive. (The proof constructs an appropriate pte-mechanism by a flipping argument, applied to the minimal convex subpolygon enclosing the chosen edge.) By adding several motions of $G(P)$ one can obtain an expansive motion that is strictly expansive on *all* eligible edges, and hence, in Theorem 4.3, there is a motion that is strictly expansive on all edges $ij \notin G$. This is actually the statement that was proved in [5], in a more general setting.

5 Other Constructions

In this section we present three related results: a different representation for the ppt-polytope that is less dependent on some seemingly arbitrary choice of parameters f, and the two constructions leading to the associahedron: 2-dimensional points in convex position and the one-dimensional expansion polytope.

5.1 A redefinition of the ppt-polyhedron

Let $P = \{p_1, \ldots, p_n\}$ be a fixed point configuration in general position. As before, to each possible choice of parameters $f = (f_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ we associate the polyhedron $\bar{X}_f$ defined by the equations (2) and (7), and the polytope X_f obtained setting to equalities the inequalities corresponding to convex hull edges. The case $f \equiv 0$ produces the expansion cone, with the polytope degenerating to a point, and the choices $f_{ij} = (p_i - p_j)^2(p_i^2 + p_j^2 + p_i p_j)$ or $f_{ij} = \det(0, p_i, p_j)^2$ produce our polytope of pointed pseudo-triangulations, according to Theorem 3.9. But the results of Section 3 imply that actually any other choice of f_{ij}'s would provide (combinatorially) the same polyhedron and polytope as long as it satisfies equation (9) for every four points p_{i_1}, p_{i_2}, p_{i_3} and p_{i_4}, where the ω_{ij}'s are the self-stress on the four points with sign chosen as in Lemma 2.4.

In this section we give a new construction for the same polyhedron of Section 3, with the advantage that it does not depend on any choice of parameters. It has the disadvantage, however, that it involves many more variables: one for each of the $\binom{n}{2}$ possible edges among the n points.

A change in a particular f_{ij} amounts geometrically to a translation of the corresponding facet of our polyhedron. We will call two choices f and f' *equivalent* if there is a global translation in $\mathbb{R}^{2n}$ which transforms each inequality defining $\bar{X}_f$ to the corresponding one defining $\bar{X}_{f'}$. This is stronger than requiring the translation to send $\bar{X}_f$ to $\bar{X}_{f'}$, because some inequalities might be redundant.

The next lemma tells us when two choices are equivalent. For example, the choices f and $2f'$ of Theorem 3.9 are equivalent.

Lemma 5.1. (Equivalence of parameters) *Let $f = (f_{ij})$ and $f' = (f'_{ij})$ be two vectors in $\mathbb{R}^{\binom{n}{2}}$. Then, the following properties are equivalent:*

(a) The choices of parameters f and f' are equivalent. In particular, $\bar{X}_f$ and $\bar{X}_{f'}$ are translates.

(b) $f - f' \in \operatorname{Im} M$, where $M\colon \mathbb{R}^{2n} \to \mathbb{R}^{\binom{n}{2}}$ is the rigidity map for the complete graph on P.

(c) For any four points p_{i_1}, p_{i_2}, p_{i_3} and p_{i_4} in P one has

$$\sum_{i<j\in\{i_1,i_2,i_3,i_4\}} \omega_{ij} f_{ij} = \sum_{i<j\in\{i_1,i_2,i_3,i_4\}} \omega_{ij} f'_{ij}$$

where the ω_{ij}'s are a nonzero self-stress of the complete graph on those four points.

Proof. Let us first prove (b)⇒(a). Suppose that $f - f'$ equals $M(a_1, \ldots, a_n)$ for a certain infinitesimal motion $a = (a_1, \ldots, a_n) \in \mathbb{R}^{2n}$. Then a translation of the vector a to the halfspace $\langle v_j - v_i, p_i - p_j\rangle \geq f'_{ij}$ produces the halfspace $\langle v_j - a_j - v_i + a_i, p_i - p_j\rangle \geq f'_{ij}$, or, equivalently, $\langle v_j - v_i, p_i - p_j\rangle \geq f'_{ij} + \langle a_j - a_i, p_i - p_j\rangle = f_{ij}$.

The proof of the other direction follows the same idea. The two choices being equivalent implies that the two inequalities

$$\langle v_j - v_i, p_i - p_j\rangle \geq f'_{ij} \quad \text{and} \quad \langle v_j + a_j - (v_i + a_i), p_i - p_j\rangle \geq f_{ij}$$

are equivalent for all v_i and v_j. This means that $\langle a_j - a_i, p_i - p_j\rangle = f_{ij} - f'_{ij}$ holds for every ij. Hence $f - f' = M(a)$.

The implication (b)⇒(c) follows directly from one direction of Lemma 2.3. For (c)⇒(b) we will assume without loss of generality that $f' = 0$. For example, let $f'' = f - f'$ and then the implication (c)⇒(b) for the vectors f'' and 0 gives it also for f and f'.

Lemma 2.3 actually gives (c)⇒(b) "for each quadruple": if (c) holds, then for each four points there is an infinitesimal motion $(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4})$ whose image by the rigidity map of the four points are the six relevant entries of $f - f'$. The motion for a quadruple is not unique, but any two choices differ only by a trivial motion of the quadruple.

To define a global motion $(a_1, \ldots, a_n)$ of the whole configuration, let us start by setting $a_1 = (0,0)$ and $a_2 = (0,b)$, where b must be the unique number satisfying $\langle p_1 - p_2, (0,b)\rangle = f_{12}$. (We assume without loss of generality that p_1 and p_2 do not have the same y-coordinate.) The condition $f = M(a)$ on the edges $1i$ and $2i$ then uniquely defines a_i for every $i \neq 1,2$, because these two equations are linearly independent, the directions $p_i - p_1$ and $p_i - p_2$ being not parallel. To see that this gobal motion satisfies $f = M(a)$ also for any other edge ij ($i \neq 1,2$, $j \neq 1,2$), it is sufficient to consider the quadruple (p_1, p_2, p_i, p_j). By construction, the motion (a_1, a_2, a_i, a_j) satisfies the condition $f = M(a)$ for five of the six edges in the quadruple. Assumption (c) for $f' = 0$ says that $\sum_{k,l \in \{1,2,i,j\}} \omega_{kl} f_{ij} = 0$ which, by Lemma 2.3, implies that f restricted to (p_1, p_2, p_i, p_j) is the set of edge increases produced by some motion $v = (v_1, v_2, v_i, v_j)$. This motion can be normalized to $v_1 = (0,0)$ and $v_2 = (0,b)$ and then it must coincide with a by our uniqueness argument above. □

Lemma 5.1 tells us in particular that the space of all parameters equivalent to a certain $f \in \mathbb{R}^{\binom{n}{2}}$ equals the affine subspace containing f and parallel to $\operatorname{Im} M$, where M is the rigidity map of the complete graph on our point set. In particular, the map $g \mapsto f + g$ identifies bijectively each translation g on $\operatorname{Im} M$ with a choice of parameters $f + g$ equivalent to f. Since the complete graph is rigid, M is injective (after identifying motions which differ by a trivial motion) and it gives a canonical bijection between translations in $\operatorname{Im} M$ and elements of the $(2n-3)$-dimensional space of infinitesimal motions in which $\bar{X}_f$ lives. On the other hand, the lemma also tells us that the choices of parameters equivalent to f are those satisfying the equations in part (c).

All this, together with the obvious fact that any subset S of a vector space V is linearly isomorphic to the *space of translations $v \in V$ for which $S + v$ contains the origin* implies the following:

Theorem 5.2. *For any $f \in \mathbb{R}^{\binom{n}{2}}$, the polyhedron $\bar{X}_f$ is linearly isomorphic to the one defined in $\mathbb{R}^{\binom{n}{2}}$ by the following $\binom{n}{4}$ equations and $\binom{n}{2}$ inequalities. (In these equations, $d = (d_{ij})_{ij \in \binom{n}{2}}$ represents the set of variables):*

$$\sum_{i<j \in \{i_1,i_2,i_3,i_4\}} \omega_{ij} f_{ij} = \sum_{i<j \in \{i_1,i_2,i_3,i_4\}} \omega_{ij} d_{ij}, \qquad \forall\, i_1, i_2, i_3, i_4 \in \{1, \ldots, n\},$$

$$d_{ij} \leq 0, \qquad \forall\, i, j \in \{1, \ldots, n\}.$$

Moreover, setting to equalities the inequalities corresponding to convex hull edges produces a polytope linearly isomorphic to X_f.

Proof. The $\binom{n}{4}$ equalities express the fact that d is equivalent to f. The inequalities $d_{ij} \leq 0$ (or $= 0$ for some of them, in the polyotpe case) express the fact that $\bar{X}_d$ (or X_d) contains the origin. □

Taking into account that $\sum \omega_{ij} f'_{ij} = 1$ for any four points, for the valid choice f' of Theorem 3.9, we conclude that:

Corollary 5.3. *For any given point set $P = \{p_1, \ldots, p_n\}$ in the plane in general position, the following $\binom{n}{4}$ equalities and $\binom{n}{2}$ inequalities define a simple polyhedron in $\mathbb{R}^{\binom{n}{2}}$ linearly isomorphic to that of Theorem 3.1.*

- *$\sum \omega_{ij} d_{ij} = 1$ for every quadruple, where the ω_{ij}'s of each equation are the self-stress on the corresponding quadruple stated in Lemma 2.4.*

- $d_{ij} \leq 0$ *for every variable.*

The maximal bounded face in the polyhedron is obtained setting to equalities the inequalities corresponding to convex hull edges. □

The $\binom{n}{4}$ equations are of course highly redundant. It follows from the proof of Lemma 5.1 that the $\binom{n-2}{2}$ quadruples involving two fixed vertices are sufficient. Subtracting this number from the number $\binom{n}{2}$ of variables actually gives the right dimension $2n-3$ of the polyhedron.

5.2 Convex position and the associahedron

Suppose now that our n points $P = \{p_1, \ldots, p_n\}$ are in (ordered) convex position. Here and in nthe sequel all indices are regarded modulo n. In Section 2 we noticed that the polytope of pointed pseudo-triangulations of P is combinatorially the same thing as the secondary polytope, which for a convex n-gon is an associahedron. We prove here that, in fact, the secondary polytope and the ppt-polytope are affinely isomorphic.

The first problem we encounter is that so far we have only facet descriptions for the ppt-polytope, while the secondary polytope is defined by the coordinates of its vertices. We recall that the secondary polytope lives in $\mathbb{R}^n$ and that the i-th coordinate of the vertex corresponding to a certain triangulation T equals the total area of all triangles of T incident to p_i. Denote this area as $\mathrm{Area}_T(p_i)$. For convenience we will work with a normalized definition of area of a triangle with vertices p, q and r as being equal to $|\det(p,q,r)|$. We also assume our points to be ordered counter-clockwise, so that $\det(p_i, p_j, p_k)$ is positive if and only if i, j and k appear in this order in the cyclic ordering of $\{1, \ldots, n\}$. In this way

$$\mathrm{Area}_T(p_i) := \sum_{l=1}^{t-1} \det(p_i, p_{j_l}, p_{j_{l+1}})$$

where $\{p_{j_1}, \ldots, p_{j_t}\}$ is the ordered sequence of vertices adjacent to p_i in T.

Our first task is to compute the coordinates of the vertices of the ppt-polytope. Notice that by definition the coordinates corresponding to edges of T are zero, since the inequalities corresponding to the edges of T are satisfied with equality. It will turn out that we do not need all the coordinates, but only those corresponding to *almost-convex-hull edges* $p_{i-1}p_{i+1}$.

Lemma 5.4. *Let T be a triangulation of P. Then, in the ppt-polytope of Corollary 5.3, the coordinate $d_{i-1,i+1}$ corresponding to an almost-convex-hull edge equals*

$$d_{i-1,i+1} = -\det(p_{i-1}, p_i, p_{i+1}) \left(\mathrm{Area}_T(p_i) - \det(p_{i-1}, p_i, p_{i+1})\right).$$

Proof. Let $p_{j_1}, \ldots, p_{j_t}$ be the ordered sequence of vertices adjacent to p_i in T, with $p_{j_1} = p_{i+1}$ and $p_{j_t} = p_{i-1}$. We will prove by induction on $k = 2, \ldots, t$ that the coordinate $d_{j_1 j_k}$ of the ppt-polytope vertex corresponding to T equals

$$d_{j_1 j_k} = -\det(p_i, p_{j_1}, p_{j_k}) \,\mathrm{Area}(p_{j_1}, \ldots, p_{j_k})$$

where $\mathrm{Area}(p_{j_1}, \ldots, p_{j_k})$ denotes the area of the polygon with vertices $p_{j_1}, \ldots, p_{j_k}$, in that order. This reduces to the formula in the statement for $k = t$.

The base case $k = 2$ is trivial: since $j_1 j_2$ is an edge in the triangulation, we have $d_{j_1 j_2} = 0$. To compute $d_{j_1 j_k}$ for $k > 2$ we consider the quadruple $p_i, p_{j_1}, p_{j_{k-1}}, p_{j_k}$. The only non-zero d's on this quadruple are $d_{j_1 j_{k-1}}$ and $d_{j_1 j_k}$. (For $k = 3$, $d_{j_1 j_{k-1}}$ is also zero.) Hence, the equation $\sum \omega_{\alpha\beta} d_{\alpha\beta} = 1$ for this quadruple reduces to

$$d_{j_1 j_k} \omega_{j_1 j_k} + d_{j_1 j_{k-1}} \omega_{j_1 j_{k-1}} = 1.$$

From this we infer the stated value for $d_{j_1 j_k}$ from the known values of the other quantities:

$$\begin{aligned}\omega_{j_1 j_k} &= \frac{1}{\det(p_i, p_{j_1}, p_{j_k}) \det(p_{j_1}, p_{j_{k-1}}, p_{j_k})}, \\ \omega_{j_1 j_{k-1}} &= \frac{-1}{\det(p_i, p_{j_1}, p_{j_{k-1}}) \det(p_{j_1}, p_{j_{k-1}}, p_{j_k})}, \text{ and} \\ d_{j_1 j_{k-1}} &= -\det(p_i, p_{j_1}, p_{j_{k-1}}) \operatorname{Area}(p_{j_1}, \dots, p_{j_{k-1}}).\end{aligned}$$

(The last equation is the inductive hypothesis.) □

This Lemma immediately implies the following:

Corollary 5.5. *The affine map* $a_i = -\frac{d_{i-1,i+1}}{\det(p_{i-1},p_i,p_{i+1})} + \det(p_{i-1}, p_i, p_{i+1})$ *gives the secondary-polytope coordinates* $(a_1, \dots, a_n)$ *of a triangulation* T *from the coordinates* $(d_{ij})_{ij \in \binom{n}{2}}$ *of* T *in the ppt-polytope of Corollary 5.3.* □

Observe that Corollary 5.5 implies that, for points in convex position, we can consider the ppt-polytope of Corollary 5.3 as lying in the n-dimensional space given by the coordinates $d_{i-1,i+1}$. For the polyhedron, we additionally need the coordinates $d_{i,i+1}$. (These are zero on the polytope, but not on the polyhedron.)

The following translates Corollary 5.5 into the original definition of the polytope of pointed pseudo-triangulations contained in Theorem 3.1:

Proposition 5.6. *Let* $v = (v_1, \dots, v_n)$ *be the coordinates of a triangulation* T *in the polytope* X_f *of Theorem 3.1, with* f *satisfying the equations* $\sum \omega_{ij} f_{ij} = 1$ *for every quadruple. Let* $(\delta_{ij})_{i,j \in \binom{n}{2}} = M(v)$*, where* M *is the rigidity map of the complete graph. Then:*

1. *The coordinates of* T *in the ppt-polytope of Corollary 5.3 are*

$$d_{ij} = f_{ij} - \delta_{ij}.$$

2. *The following transformation sends* v *to the coordinates of* T *in the secondary polytope:*

$$\begin{aligned}a_i(v) &:= -\frac{d_{i-1,i+1}}{\det(p_{i-1}, p_i, p_{i+1})} + \det(p_{i-1}, p_i, p_{i+1}) \\ &= \frac{\langle p_{i+1} - p_{i-1}, v_{i+1} - v_{i-1} \rangle - f_{i-1,i+1}}{\det(p_{i-1}, p_i, p_{i+1})} + \det(p_{i-1}, p_i, p_{i+1}).\end{aligned}$$

Proof. For part 1, just observe that the (unique) motion which expands each edge of T with edge length increase $f_{ij} - \delta_{ij}$ is clearly $v - v = 0$. Hence, subtracting δ_{ij} from each f_{ij} amounts to translate X_f to a position where the vertex corresponding to T is at the origin. The perturbation parameters of this translated polytope are, by definition, the coordinates of the vertex corresponding to T in the polytope of Corollary 5.3. Part 2 is clear, from Corollary 5.5. □

5.3 The 1-Dimensional Case: the Associahedron Again

By considering 1-dimensional expansive motions, in this section we will recover the associahedron via a different route. The analogy of this construction to the 2-dimensional case will become even more apparent in Section 6.

The polytope of constrained expansions in dimension 1. In the 1-dimensional case we will rewrite equations (7) as

$$v_j - v_i \geq g_{ij}, \qquad \forall i < j. \tag{12}$$

One set of inequalities is equivalent to the other under the change of constants $g_{ij}(p_j - p_i) = f_{ij}$, for any $i < j$. This reformulation explicitly shows that the solution set does not depend on the point set $P = \{p_1, \ldots, p_n\}$ that we choose. We denote this solution set $\bar{X}_g$, to mimic the notation of the 2-dimensional case.

It is easy to see that the polyhedron $\bar{X}_g$ is full-dimensional and it contains no lines if we add the normalization equation $v_1 = 0$. Hence, after normalization, it has dimension $n-1$ and contains some vertex. For any vertex v or for any feasible point $v \in \bar{X}_g$, we may look at the set $E(v)$ of tight inequalities at v:

$$E(v) := \{\, ij \mid 1 \leq i < j \leq n, v_j - v_i = g_{ij} \,\}$$

We regard $E(v)$ as the set of edges of a graph on the vertices $\{1, \ldots, n\}$.

One may get various polyhedra by choosing different numbers g_{ij} in (7). We choose them with the following properties.

$$g_{il} + g_{jk} > g_{ik} + g_{jl}, \qquad \forall 1 \leq i < j \leq k < l \leq n. \tag{13}$$

For $j = k$ we use this with the interpretation $g_{jj} = 0$, so we require

$$g_{il} > g_{ik} + g_{kl}, \qquad \forall 1 \leq i < k < l \leq n. \tag{14}$$

One way to satisfy these conditions is to select

$$g_{ij} := h(t_j - t_i), \quad \forall i < j \tag{15}$$

for an arbitrary strictly convex function h with $h(0) = 0$ and arbitrary real numbers $t_1 < \cdots < t_n$. The simplest choice is $h(t) = t^2$ and $t_i = i$, yielding $g_{ij} = (i-j)^2$.

Two edges ij and jk with $i < j < k$ are called *transitive edges*, and edges ik and jl with $i < j < k < l$ are called *crossing edges*.

Lemma 5.7. *If g satisfies (13–14) and $v \in \bar{X}_g$, then $E(v)$ cannot contain transitive or crossing edges.*

Proof. If we have two transitive edges $ij, jk \in E(v)$ this means that $v_j - v_i = g_{ij}$ and $v_k - v_j = g_{jk}$. This gives $v_k - v_i = g_{ij} + g_{jk} < g_{ik}$, by (14), and thus v cannot be in $\bar{X}_g$ because it violates (7). The other statement follows similarly. ☐

Non-crossing alternating trees. A graph without transitive edges is called an *alternating* or *intransitive* graph: every path in an alternating graph changes continually between up and down.

Lemma 5.8. *A graph on the vertex set $\{1, \ldots, n\}$ without transitive or crossing edges cannot contain a cycle.*

Proof. Assume that C is a cycle without transitive edges. Let i and m be the lowest and the highest-numbered vertex of a cycle C, and let ik be an edge of C incident to i, but different from im. The next vertex on the cycle after k must be between i and k; continuing the cycle, we must eventually reach m, so there must be an edge jl which jumps over k, and we have a pair ik, jl of crossing edges. ☐

Since the polyhedron is $(n-1)$-dimensional, the set $E(v)$ of a vertex v must contain at least $n-1$ edges. We have just seen that it is acyclic, and hence it must be a tree and contain exactly $n-1$ edges. So we get:

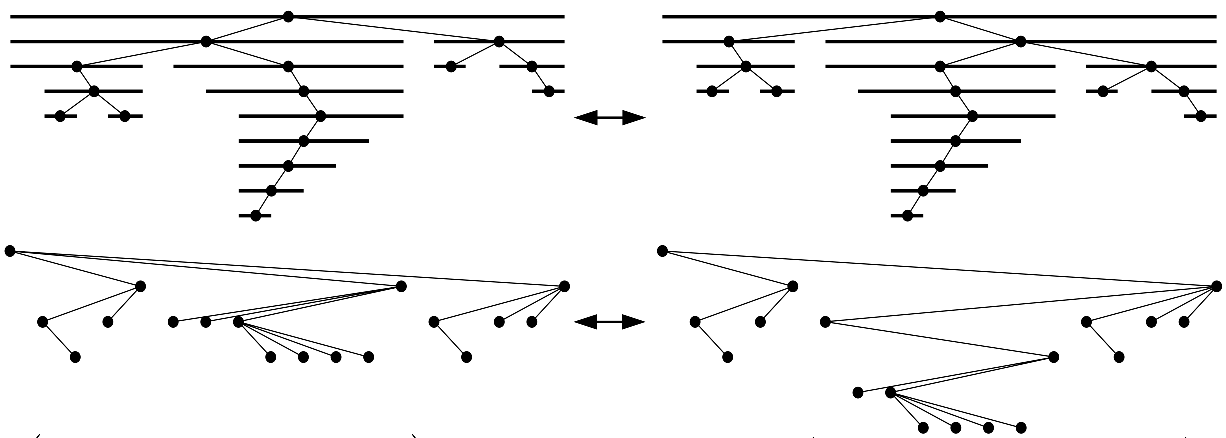
Proposition 5.9. *$\bar{X}_g$ is a simple polyhedron. The tight inequalities for each vertex correspond to non-crossing alternating trees.* ☐

We will see below that $\bar{X}_g$ contains in fact *all* non-crossing alternating trees as vertices.

A new realization of the associahedron. Let's look at the combinatorial properties of these trees. Alternating trees have been studied in combinatorics in several papers, see for example [17, 18] or [19, Exercise 5.41, pp. 90–92] and the references given there. Non-crossing alternating trees were studied only by Gelfand, Graev, and Postnikov, under the name of "standard trees". They proved the following fact [8, Theorem 6.4].

Proposition 5.10. *The non-crossing alternating trees non $n+1$ points are in one-to-one correspondence with the binary trees on n vertices, and hence their number is the n-th Catalan number $\binom{2n}{n}/(n+1)$.* $\square$

The bijection given in [8] to prove this fact is that the vertices of the binary tree correspond to the edges of the alternating tree. It is easy to see that every non-crossing alternating tree must contain the edge $1n$. Removing this edge splits the tree into two parts; they correspond to the two subtrees of the root in the binary tree. The two parts are handled recursively. Even simpler is the bijection to bracketings (ways to insert $n-1$ pairs of parentheses in a string of n letters). Just change edge ij by a parenthesis enclosing the i-th and j-th letter.



(((a((bc)(de))) (f(g(((((hi)j)k)l)m))))((no)(p(qr)))) $\longleftrightarrow$ ((a((bc)(de)))((f(g(((((hi)j)k)l)m))) ((no)(p(qr)))))

Figure 5: The bijection between non-crossing binary trees (up), alternating trees (middle) and bracketings (bottom). The flipping operation in each case is shown.

Figure 5 gives an example of these correspondences, including the correspondence of the respective flipping operations: If we remove any edge $e \neq 1n$ from a non-crossing alternating tree T, there is precisely one other non-crossing alternating tree T' which shares the edges $T - \{e\}$ with T. This exchange operation corresponds under the above bijection to a tree rotation of the binary tree, and to a single application of the associative law (remove a pair of interior parentheses and insert another one in the only possible way) in a bracketing.

Observe that still we don't know that *all* the non-crossing and alternating trees appear as vertices of X_g. But this is easy to prove: since X_g is simple of dimension $d-2$, its graph is regular of degree $d-2$. And it is a subgraph of the graph of rotations between binary trees, which is also $(d-2)$-regular and connected. Hence, the two graphs coincide.

If we now look at faces of $\bar{X}_g$, rather than vertices, the tight edges for each of them form a non-crossing alternating *forest*. Such a forest G is an expansive mechanism if and only if the edge $1n$ is not in G: If that edge is present, p_1 and p_n are fixed and any other p_i must approach one of the two. If the edge $1n$ is not present, then let i be the maximum index for which $1i$ is present. Then the motion $v_1 = \cdots = v_i = 0 < v_{i+1} = \cdots = v_n$ is an expansive flex of G.

Hence, $\bar{X}_g$ has a unique maximal bounded face, the facet given by the equation $v_n - v_1 = g_{1n}$ and corresponding to the edge $1n$ alone. This facet is then an $(n-2)$-dimensional simple polytope that we denote X_g. The $n-2$ neighbors of each vertex v correspond to the $n-2$ possible exchanges of edges different from $1n$ in the tree $E(v)$.

Theorem 5.11. *X_g is a simple $(n-2)$-dimensional polytope whose face poset is that of the associahedron. Vertices are in one-to-one correspondence with the non-crossing alternating trees on n vertices. Two vertices are adjacent if and only if the two non-crossing alternating trees differ in a single edge.*

$\bar{X}_g$ is an unbounded polyhedron with the same vertex set as X_g. The extreme rays correspond to the non-crossing alternating trees with the edge $1n$ removed.

Proof. Only the statement regarding the face poset remains to be proved. This can be proved in two ways: On the one hand, we already know that the graphs of X_g and of the $(d-2)$-associahedron coincide (the latter being the graph of rotations between binary trees). And simple polytopes with the same graph have also the same face poset. (This is a result of Blind and Mani; see [22, Section 3.4]). As a second proof, the correspondence between non-crossing alternating trees and bracketings trivially extends to a correspondence between non-crossing alternating forests containing $1n$ and "partial bracketings" in a string of n letters which include the parentheses enclosing the whole string. The poset of such things is the face poset of the associahedron (see [19], Proposition 6.2.1 and Exercise 6.33). In particular, the face-poset of X_g is a subposet of the face-poset of the associahedron, and two polytopes of the same dimension cannot have their face posets properly contained in one another. This is true in general by topological reasons, but specially obvious in our case since we know our polytopes to be simple and their vertices to correspond one to one. Each vertex is incident to exactly $\binom{d-2}{i}$ faces of dimension i in both polytopes, and two subsets of cardinality $\binom{d-2}{i}$ cannot be properly contained in one another. □

A result which is related to Theorem 5.11 was proved by Gelfand, Graev, and Postnikov [8, Theorem 6.3], in a setting dual to ours: there a triangulation of a certain polytope was constructed. The non-crossing alternating trees correspond to the *simplices* of the triangulation. It is shown explicitly that the simplices form a partition of the polytope. Certain numbers g_{ij} are then associated to the *vertices* of the polytope to show that the triangulation is a projection of the boundary of a higher-dimensional polytope. Incidentally, the numbers that were suggested for this purpose are $(i-j)^2$, which coincides with the simple proposal given above after (15), but the calculations are not given in the paper [8].

One easily sees that the conditions (13–14) on g are also necessary for the theorem to hold: If any of these conditions would hold as an equality or as an inequality in the opposite direction, the argument of Lemma 5.7 would work in the opposite direction, and certain non-crossing alternating trees would be excluded. Thus, (13–14) gives a complete characterization of the "valid" parameter values g_{ij}.

Further Remarks. The result we presented in this section is surprising in two ways: first, that it produces such a well studied object as the associahedron; second, that it requires additional types of linear constraints that are not needed in dimension 2. Indeed, equations (14) in the 1-dimensional case are the exact analogue of equations (9) in the 2-dimensional case, as we will see in Section 6. But Equations (13) have no analogy. This second aspect makes the task of producing 3d generalization of the constructions of this paper more challenging, as it does not seem to be a straightforward pattern for producing the linear constraints whose instantiations in 1d and 2d give the polytopes of expansive motions.

The conditions (13–14) leave a lot of freedom for the choice of the variables g_{ij}. We have an $\binom{n}{2}$-dimensional parameter space. This is in contrast to the classical representation, which has $2n$ parameters (the coordinates of n points in the plane). If we select the parameters to be integral, we obtain an associahedron which is an integral polytope (this is also true for the classical associahedron for a polygon with integer vertices.) But observe that, in fact, the associahedra obtained here are in a sense much more special than the classical associahedra obtained as secondary polytopes. They have $n-2$ pairs of parallel facets, given by the equations $v_i = v_1 + g_{1i}$ and $v_i = v_n - g_{in}$ (i.e., corresponding to the pairs of edges $1i$ and in). This is no surprise, since we have constructed our polytope by perturbing (a region of) a Coxeter arrangement, whose hyperplanes are in extremely non-general position.

These parallel facets also prove that we get associahedra which are not affinely equivalent to the classical ones: one can check that classical associahedra have no parallel facets. It is however conceivable that they are related by projective transformations.

Relations to optimization and the Monge matrices. Constraints of the form (12) are commonly found in project planning and critical path analysis. The variables v_i represent unknown start times of tasks, and the constraints (12) specify waiting conditions between different tasks. For example, the minimization of $v_n - v_1$ is a longest path problem in an acyclic network. The different vertices of X_g correspond to the different optimal solutions when we apply various linear objective functions.

Matrices $G = (g_{ij})$ with the property (13) are said to have the *Monge property*, if we set $g_{ii} = 0$ and $g_{ij} = \infty$ for $i > j$. The Monge property has received a great deal of attention in optimization because it arises often in applications and it characterizes special classes of optimization problems that can be solved efficiently, see [4] for a survey.

The dual linear program of the linear programming problem (12) (with a suitable objective function) is a minimum-cost flow problem on an acyclic network with edges ij for $1 \leq i < j \leq n$, and cost coefficients $-g_{ij}$. Network flow is actually one of the oldest areas in optimization in which the Monge property has been applied, and where it has been shown that optimal solutions can be obtained by a greedy algorithm in certain cases. The non-crossing alternating trees are just the different possible subgraphs of those edges which carry flow in an optimal solution.

6 Towards a General Framework, in Arbitrary Dimension

To make more explicit the analogy between our 1d and 2d constructions we'll rewrite the 1d construction back in terms of equations (7) instead of (12). As we mentioned at the beginning of Section 5.3, the way to do this is to substitute $g_{i,j} = f_{i,j}/(p_j - p_i)$ everywhere. In particular, equations (14) become

$$\frac{f_{ik}}{p_i - p_k} + \frac{f_{kl}}{p_k - p_l} + \frac{f_{il}}{p_l - p_i} > 0.$$

But $\omega_{ik} = \frac{1}{p_i - p_k}$, $\omega_{kl} = \frac{1}{p_k - p_l}$ and $\omega_{il} = \frac{1}{p_l - p_i}$ define a self-stress on any 1-dimensional point set $\{p_i, p_k, p_l\}$. Hence, equations (14) are the exact analogue of equations (9) of the 2d case. The difference is that in 2d these equations are necessary and sufficient for a choice of parameters to be "valid", while in 1d we need the additional equations (13), which do not follow from (14) as the following example shows:

$$g_{12} = g_{23} = g_{34} = 1, \qquad g_{13} = g_{24} = 2.2, \qquad g_{14} = 3.3.$$

The main remaining question is whether the fact that both in 1d and 2d there are choices of perturbation parameters providing polytopes of constrained expansions with nice combinatorial properties is just a coincidence. This question is related to whether there is a sensible translation of the definition of *non-crossing and pointed* graphs to 3d. Some of the properties that we would like to obtain are:

- The poset of such objects should be isomorphic to some (hopefully simple) polyhedron in the $(3n-6)$-dimensional space of non-trivial 3d motions obtained by translating the defining hyperplanes of the cone of expansive motions.
- The maximal objects should be some of the minimally rigid graphs on the point set.
- Ideally, the polyhedron should have a unique maximal bounded face.

What seems to be the most difficult thing to obtain is the uniqueness of the maximal bounded face (as well as the fact that we want good combinatorial characterization and properties of the minimally rigid graphs involved). Any generic choice of perturbation parameters provides a polyhedron with the other two properties. One difficulty is that the combinatorics of minimally rigid graphs in dimension three is not fully understood. Perhaps a good case to start with would be points in convex and general position in dimension 3, for which the convex hull edges provide a very canonical minimally rigid graph.

It would also be good to have a generalization of our construction to 2d point sets which are not in general position. If we use the 2d definition of f from equations (10) with many points on a common line, there are solutions in which all the inequalities are tight. In this way we get essentially the one-dimensional expansion cone of Proposition 2.5, when we project all vectors v_i on the direction of that line. (This is the same situation as when all relations (14) are satisfied as equations.) One way to get rid of this degeneracy is to "perturb the perturbations" by adding an infinitesimal component of the one-dimensional expansion parameters of Section 5.3: instead of f we use $f_{ij}^{(2)} + \varepsilon f_{ij}^{(1)}$, where $f^{(2)}$ is given by (10), $f_{ij}^{(1)}$ satisfy the restrictions required for the 1d perturbation parameters, and $\varepsilon > 0$ is sufficiently small. We have not further investigated this idea.

Another consideration which may help to treat degenerate point sets and arbitrary dimensional ones is the following. For an arbitrary point set $P = \{p_1, \dots, p_n\} \in \mathbb{R}^d$ we can consider its cone of expansive motions $\bar{X}_0(P)$ and for any choice of parameters $f \in \mathbb{R}^{\binom{n}{2}}$ the corresponding polyhedron $\bar{X}_f(P)$ of constrained expansions, which has $\bar{X}_0(P)$ as its recession cone. The equivalence of parts (a) and (b) of Lemma 5.1 remains valid and with the same proof. But what is the analogue of part (c)?

Remember that a *circuit* of a point set $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$ is a minimal affinely dependent subset. If P is in general position, its set of circuits $\mathcal{C}(P)$ is $\binom{P}{d+2}$. In general, any set of $d+2$ points spanning d dimensions contains a unique circuit, but this circuit can have less than $d+2$ elements. In any case, the complete graph on a circuit has the property that the removal of any single edge provides a minimally rigid framework. In particular, each circuit C has (up to a constant factor) a unique self stress $\{\omega_{ij}\}_{i,j\in C}$ with no vanishing ω_{ij}.

Let us consider the following linear maps:

$$\begin{array}{rccc} M\colon & \mathbb{R}^{dn} & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ & (v_1, \dots, v_n) & \longmapsto & \langle v_i - v_j, p_i - p_j \rangle, \\ \Delta\colon & \mathbb{R}^{\binom{n}{2}} & \longrightarrow & \mathbb{R}^{\mathcal{C}(P)} \\ & \{f_{ij}\}_{ij\in\binom{n}{2}} & \longmapsto & \Big\{ \sum\limits_{ij\in C} \omega_{ij} f_{ij} \Big\}_{C\in\mathcal{C}(P)}. \end{array}$$

M is just the rigidity map of the complete graph on P. The equivalence of (b) and (c) in Lemma 5.1 can be rephrased as

$$\ker \Delta = \operatorname{Im} M, \text{ for every planar point set in general position.}$$

But in fact the same is true in arbitrary dimension and under much weaker assumptions than general position:

Proposition 6.1. *For any point set $P \subset \mathbb{R}^d$, $\operatorname{Im} M \subseteq \ker \Delta$. If P has d affinely independent points whose spanned hyperplane contains no other point of P, then the reversed inclusion also holds.*

Proof. That $\operatorname{Im} M \subset \ker \Delta$ (i.e., $\Delta \circ M = 0$) is one direction of Lemma 2.3. To prove $\ker \Delta \subset \operatorname{Im} M$, let $f \in \mathbb{R}^{\binom{n}{2}}$ be such that $\sum_{ij\in C} \omega_{ij} f_{ij} = 0$ for every circuit C. We want to find a motion $a = (a_1, \dots, a_n)$ for which $M(a) = f$. Without loss of generality assume that $p_1, \dots, p_d$ are the claimed affinely independent points. Let us fix a motion $(a_1, \dots, a_d)$ for the first d points satisfying the $\binom{d}{2}$ equations of the system $Ma = f$ concerning these points. This motion exists and is unique up to an arbitrary translational and rotational component, because the complete graph on any affinely independent point set is minimally infinitesimally rigid.

By assumption, $\{p_1, \dots, p_d, p_i\}$ is an affine basis for every $i > d$, and hence the complete graph on it is again minimally infinitesimally rigid. Thus, there is a motion $(a'_1, \dots, a'_d, a_i)$ satisfying the $\binom{d+1}{2}$ corresponding equations of the system $Ma = f$. Adding translations and rotations we can assume $a'_j = a_j$, $j = 1, \dots, d$. So we have constructed a motion $(a_1, \dots, a_n)$ which restricts to the one chosen for the d first points and which satisfies all the equations for pairs of points that include one of the first d.

It remains to show that the equations are also satisfied for the edges kl with $k, l > d$. We follow the same idea as in the proof of Lemma 5.1. Our assumption on the point set implies that the unique circuit C contained in $\{p_1, \dots, p_d, p_k, p_l\}$ uses both p_k and p_l. To simplify notation, assume that this circuit is $\{p_1, \dots, p_i, p_k, p_l\}$. By Lemma 2.3, there is a feasible motion $(a'_1, \dots, a'_i, a'_k, a'_l)$. By translations and rotations we assume $a_j = a'_j$ for $j = 1, \dots, i$. Observe now the value of $\langle v - u, p_{j_1} - p_{j_2} \rangle$, for any of the points in the circuit and for any vectors v and u, depends only on the projection of v and u to the affine subspace spanned by C. Since the complete graphs on $\{p_1, \dots, p_i\}$, on $\{p_1, \dots, p_i, p_k\}$ and on $\{p_1, \dots, p_i, p_l\}$ are minimally infinitesimally rigid *when motions are restricted to that subspace*, we conclude that the projections of a_k and a_l to that affine subspace coincide with the projections of a'_k and a'_l. In particular, $\langle a_l - a_k, p_l - p_k \rangle = \langle a'_l - a'_k, p_l - p_k \rangle = f_{kl}$. □

Hence, Lemma 5.1 and its corollary, Theorem 5.2, hold in this generalized setting, with one equation per circuit instead of one equation per 4-tuple. The weakened general position assumption for d of the points holds for every planar point set, since, by Sylvester's theorem, any finite set of points in the plane, not all on a single line, has a line passing through only two of the points.

In dimension 3, however, the same is not true, and actually there are point sets for which $\operatorname{Im} M \neq \ker \Delta$. Consider the case of six points, three of them in one line and three in another, with the two lines being skew (not parallel and not crossing). These two sets of three points are the only two circuits in the point set. In particular, $\ker \Delta$ has at most codimension 2 in $\mathbb{R}^{15}$, i.e., it has dimension at least 13. On the other hand, $\operatorname{Im} M$ has at most the dimension of the reduced space of motions, $18 - 6 = 12$.

7 Final Comments

The main open questions related to this work are how to extend the constructions from dimensions 1 and 2 to 3 and higher, and how to treat subsets in special position in 2d. The expectation is that this would give a coherent definition for "pseudo-triangulations" in higher dimensions. Some ideas in this direction have been mentioned in Section 6.

It would also be interesting to see if there is a deeper reason behind the mysterious choice of parameters in Theorem 3.9. For example, a more transparent proof of the algebraic identities $R = 2$ and $S = 1$ in Lemmas 3.10–3.12. These identities are in a sense trivial, as they can be proved by expanding all expressions and canceling terms (better with the aid of computer algebra software!). We have actually been able to extend these identities to one more general class of planar graphs than just the complete graph on four vertices: to wheels (graphs of pyramids). A wheel is a cycle with an additional vertex attached to every vertex of the cycle. It is possible to define a set of stresses ω_{ij} on the edges of a wheels (drawn on an arbitrary point set in the plane in general position) by formulas that are quite similar to (3) in Lemma 2.4, such that $R = 2$ and $S = 1$ hold. This may be a hint that these indentities might be instances of a more general phenomenon.

As mentioned in the introduction, the representation of combinatorial structures as vertices of a polytope also opens the way for selecting a particular structure by optimizing some linear functional over the polytope. For example, the minimization of the objective function with coefficient vector $(|p_1|^2, \dots, |p_n|^2)$ over the secondary polytope gives the Delaunay triangulation. The opposite choice gives the further-site Delaunay triangulation. The most natural choice of objective function for the polytope of pointed pseudo-triangulations is $(p_1, \dots, p_n)$ or its opposite, i.e., we minimize or maximize $\sum_i \langle p_i, v_i \rangle$ over all constrained expansions which are tight on convex hull edges. We have checked that, for points in convex position, what we get does not coincide with the Delaunay triangulation of those points. In fact the result is invariant under affine transformations of the point set. The investigation of the properties of the pointed pseudo-triangulations that are defined in this way awaits further studies.

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