Surfaces

Lecture #19: Thursday, 4 December 1997
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1 Plane Curve Review

We first finish up the previous lecture topic of plane curves. The ellipse is used to show some of the properties of the **evolute**.

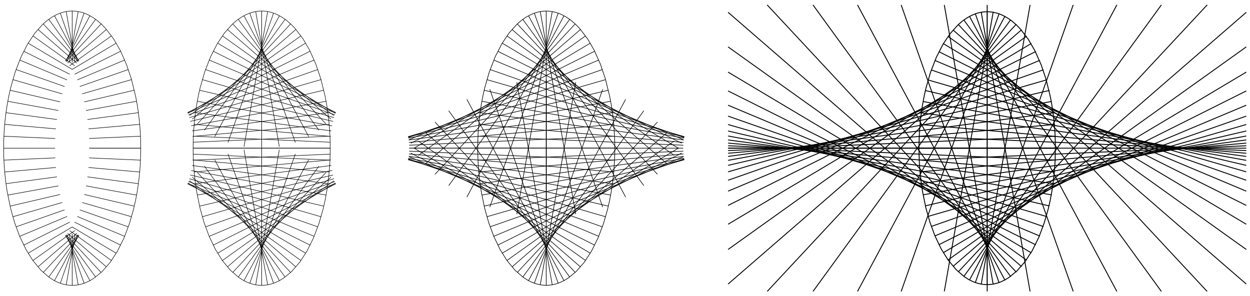

Figure 1: The envelope of the normals is the evolute.

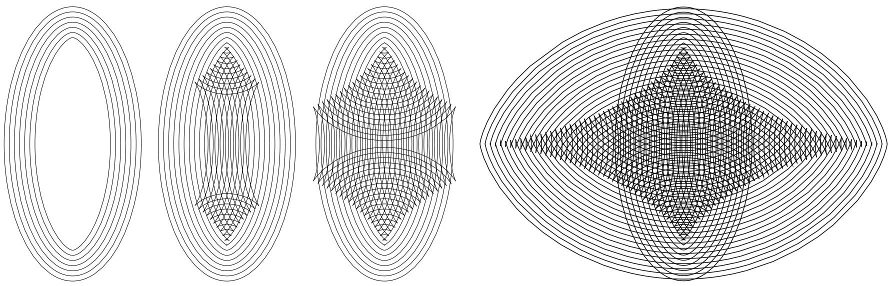

Figure 2: The evolute is the trace of wavefront singularities.

Figure 1 shows the normals to an ellipse drawn at uniform intervals. Recall that the envelope of the normals is the evolute. The figure shows that this evolute is a well defined curve which is the obviously traced out curved diamond shape. Notice that the evolute has cusps. Although it's not shown in the figure, if you think about it you can see that if you drew an osculating circle to the ellipse at any point, its center would fall on the evolute. Another way to think about this is to consider that the differential section of the surface around a point is a tiny arc of the circle centered on the evolute.

You may also recall from the previous lecture that the evolute is the trace of wavefront singularities. Figure 2 shows the offset curves of an ellipse. Again, these curves are drawn at uniform offset intervals. Note that they fold over, and their singularities and cusps lie on the evolute.

These two figures illustrate caustics, a subject we will return to in the next lecture. This one is devoted to curves in space and surfaces.

2 Space Curves

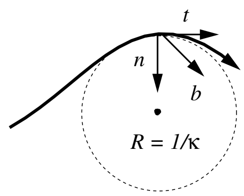

Figure 3: A space curve with a Frenet frame made from the tangent, normal, and binormal. The radius of the osculating circle is proportional to the curvature.

The previous lecture dealt with curves in the plane. Figure 3 shows a curve in space. A **space curve** can be parametrized by arc length with the function $x(s)$. At every point it has a well-defined tangent $t = x'(s)$, which is the derivative of the curve with respect to s. The derivative of the tangent is the curvature times the normal: $t' = \kappa n$. We can once again fit an osculating circle to the curve at the point, whose radius is the inverse of the curvature. In the plane this osculating circle argument was straightforward. However, the extra degree of freedom in space gives us a whole family of circles which fit the curve. We pick the one whose center is pointed to by the normal vector n.

2.1 The Frenet Frame

We know that the tangent is perpendicular to the normal. We can define a coordinate system with the addition of a third orthogonal vector, the **binormal**: $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. This triad of axes is uniquely determined by the curve and is known as the **Frenet frame**.

The Frenet frame has the following properties:

$$\mathbf{b} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{n} = \mathbf{t} \cdot \mathbf{t} = 1 \tag{1}$$

$$\mathbf{b} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{t} = \mathbf{t} \cdot \mathbf{b} = 0 \tag{2}$$

Equation 1 expresses the fact that the axes have unit length, while equation 2 is a consequence of their orthogonality. The Frenet coordinate frame changes as you move along the curve or surface – it's a function of space. We can express any vector or point with respect to this frame. It's often useful to express changes in this frame:

$$\begin{array}{rcrcrcr} \mathbf{t}' & = & c_{tt}\mathbf{t} & + & c_{tn}\mathbf{n} & + & c_{tb}\mathbf{b} \\ \mathbf{n}' & = & c_{nt}\mathbf{t} & + & c_{nn}\mathbf{n} & + & c_{nb}\mathbf{b} \\ \mathbf{b}' & = & c_{bt}\mathbf{t} & + & c_{bn}\mathbf{n} & + & c_{bb}\mathbf{b} \end{array}$$

where c_{tt} means $(t' \cdot t)$. We already know that $t' = \kappa n$, which takes care of the first line:

$$\begin{array}{rcllclcl} \mathbf{t}' & = & 0\mathbf{t} & + & \kappa\mathbf{n} & + & 0\mathbf{b} \\ \mathbf{n}' & = & (n' \cdot t)\mathbf{t} & + & (n' \cdot n)\mathbf{n} & + & (n' \cdot b)\mathbf{b} \\ \mathbf{b}' & = & (b' \cdot t)\mathbf{t} & + & (b' \cdot n)\mathbf{n} & + & (b' \cdot b)\mathbf{b} \end{array}$$

To simplify $\mathbf{b}'$, we start with the last term. We know from above that $\mathbf{b} \cdot \mathbf{b} = 1$. When we differentiate with the chain rule we find that $\frac{d}{dx}(\mathbf{b} \cdot \mathbf{b}) = \frac{d}{dx}\mathbf{b}^2 = 2\mathbf{b} \cdot \mathbf{b}' = 0$. So $\mathbf{b} \cdot \mathbf{b}' = 0$ and the last term drops away.

We attack the first term of $\mathbf{b}'$ by noting from above that $\mathbf{b} \cdot \mathbf{t} = 0$, which when differentiated gives $\mathbf{b}' \cdot \mathbf{t} + \mathbf{b} \cdot \mathbf{t}' = 0$ and thus $\mathbf{b}' \cdot \mathbf{t} = -\mathbf{b} \cdot \mathbf{t}'$. Substituting the definition of the tangent gives $-\mathbf{b} \cdot \mathbf{t}' = -\mathbf{b} \cdot \kappa\mathbf{n} = -\kappa\mathbf{b} \cdot \mathbf{n}$. Since we know that $\mathbf{b} \cdot \mathbf{n} = 0$, we have found that $\mathbf{b}' \cdot \mathbf{t} = 0$ and can drop the first term.

To find the middle term of $\mathbf{b}'$ we use two of the results from the other terms: $\mathbf{b}' \cdot \mathbf{b} = 0$ and $\mathbf{b}' \cdot \mathbf{t} = 0$. If the dot product of two vectors is 0 they are perpendicular, so $\mathbf{b}'$ is perpendicular to $\mathbf{b}$ and $\mathbf{t}$. The same is true for $\mathbf{n}$, so $\mathbf{b}'$ must be some multiple of $\mathbf{n}$. We will define the **torsion** τ of the curve by the equation $\mathbf{b}' = -\tau\mathbf{n}$. Just as the curvature κ measures the rate of change of the tangent, the torsion τ measures the rate of change of the binormal.

Our remaining unknown is $\mathbf{n}'$. We use the chain rule as above to find that $\mathbf{n}' \cdot \mathbf{n} = 0$ and drop the middle term. For the first term, we differentiate $(\mathbf{n} \cdot \mathbf{t}) = 0$ to find that $\mathbf{n}' \cdot \mathbf{t} + \mathbf{n} \cdot \mathbf{t}' = 0$. Substituting $\mathbf{t}' = \kappa\mathbf{n}$ and $\mathbf{n} \cdot \mathbf{n} = 1$ shows that $\mathbf{n}' \cdot \mathbf{t} = -\mathbf{n} \cdot \mathbf{t}' = -\mathbf{n} \cdot \kappa\mathbf{n} = -\kappa(\mathbf{n} \cdot \mathbf{n}) = -\kappa \cdot 1$. Thus the middle term simplifies to $-\kappa$. We find the last term of $\mathbf{n}'$ using the same process: $\mathbf{n}' \cdot \mathbf{b} = -\mathbf{n} \cdot \mathbf{b}' = -\mathbf{n} \cdot (-\tau\mathbf{n}) = \tau(\mathbf{n} \cdot \mathbf{n}) = \tau \cdot 1 = \tau$. We have thus found that $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$.

We have just derived what are known as the **Frenet-Serret** formulas:

$$\boxed{\begin{array}{rcccc} \mathbf{t}' & = & & \kappa\mathbf{n} & \\ \mathbf{b}' & = & & -\tau\mathbf{n} & \\ \mathbf{n}' & = & -\kappa\mathbf{t} & & +\tau\mathbf{b} \end{array}}$$

These formulas are just a system of differential equations which tell us about the motion of the Frenet frame. It is plausible that we can integrate them to recover information about the curve itself. In fact, we can completely reconstruct a curve just from curvature and torsion at every point. We state without proof the strong theorem which states this result. It is one of the many theorems in different branches of mathematics known as the Fundamental Theorem.

Theorem 1. *The Fundamental Theorem of Curves*

Given an arc length s, and two arbitrary functions $\kappa(s)$ and $\tau(s)$, we can uniquely determine the shape of curve $x(s)$.

2.2 The Darboux Vector

At the beginning of the last lecture we pointed out that there are many legitimate ways to describe a curve: algebraic, parametric, and intrinsic. The previous section was an example of the intrinsic approach.

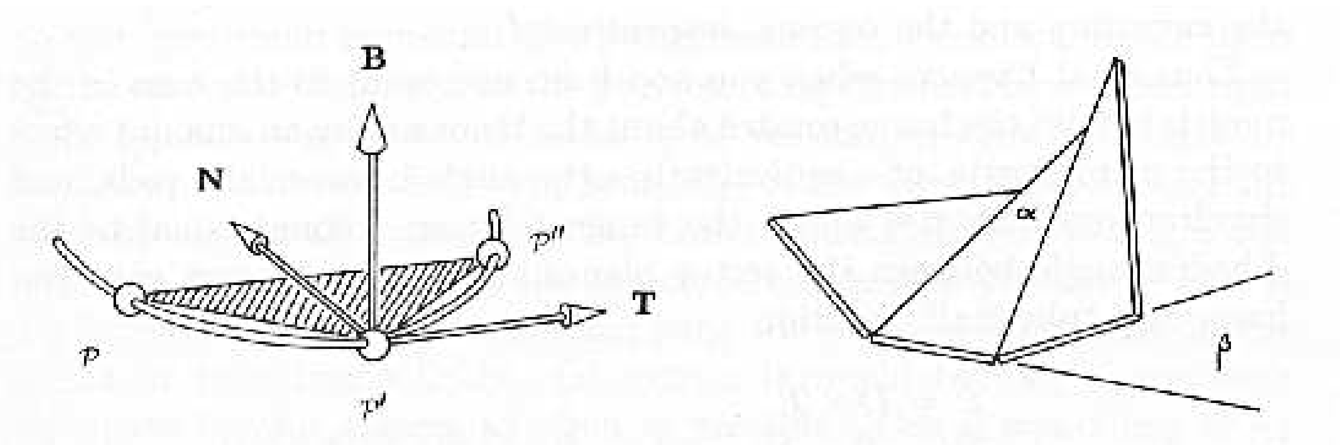

Figure 4: Two geometric interpretations of the Frenet frame from Koenderink [1]. On the left is the continuous case, while the right shows a less common piecewise linear view. Reproduced without permission from [1], Figures 110,111.

Koenderink's extremely readable book [1] builds up the ideas discussed in this lecture by describing several different interpretations in both words and pictures. Figure 4, from the book, shows two different geometrical interpretations of the Frenet frame.

On the left we can clearly see the coordinate frame defined by the normal, tangent, and binormal at point $\mathcal{P}'$. The normal lies in the shaded plane defined by the infinitesmally close points $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$. At every point there is a well defined normal – it just changes as you move.

The picture on the right shows a piecewise linear approximation to the smooth curve, as might be constructed by a set of linked rods. The rods themselves show the tangents.

The binormal for the point corresponding to $\mathcal{P}$ is drawn as an additional vertical rod. The osculating planes formed by each two neighboring rods are also shown. As the name implies, the osculating circles lie in these planes. This drawing shows that you can consider the plane as determined by two vectors instead of by a circle.

The normals at each point lie in this osculating plane, and the binormal at that point is perpendicular to this plane. The angle α is the amount of "turn" between the rods, which is the curvature. The binormal is the axis of this rotation. The angle β is the amount of turn around the tangent between two neighboring points – that is, the torsion.

Thus we can think of moving from one endpoint of a rod to the other by locally rotating the curve a little bit around the binormal and a little bit around the tangent. We thus see the motivation for defining the **Darboux vector** as that combined rotation:

$$\mathbf{d} = \kappa\mathbf{b} + \tau\mathbf{t}$$

We can use this quantity to rewrite the Fresnel-Serrat formulas concisely as

$$\begin{aligned}\mathbf{t}' &= \mathbf{d} \times \mathbf{t}\\ \mathbf{b}' &= \mathbf{d} \times \mathbf{n}\\ \mathbf{n}' &= \mathbf{d} \times \mathbf{b}\end{aligned}$$

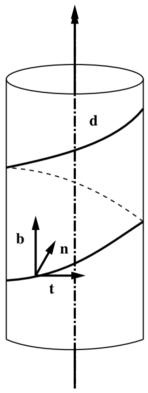

Figure 5: The Darboux vector and coordinate frame on a cylindrical helix.

It is also useful to consider the geometric interpretation of the Darboux vector and the coordinate frame on a cylindrical helix, as shown in Figure 5. The case of the cylindrical helix is unique because both the rate of translation and of rotation are constant. The Darboux vector is the axis of the cylinder, the normal points towards that axis, and the binormal is on the surface of the cylinder. The coordinate frame travels in a screw motion on a helix on the surface of the cylinder as it translates up the instantaneous tangent and rotates around the axis **d**.

3 Surfaces

We now move on from curves to surfaces. Consider a curve C passing through a point P on a surface, represented by the parametric equations u=u(t), v=v(t), so the curve in space is represented as x = x(u(t), v(t)). The tangent vector at a point is the derivative, and can be determined via the chain rule:

$$\mathbf{t} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt}$$

Thus the tangent to the curve at the point P has direction components

$$\dot{\mathbf{x}} = \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt} = \mathbf{x}_u \dot{u} + \mathbf{x}_v \dot{v}$$

The tangent vector to C at P is a linear combination of the vectors $\dot{\mathbf{x}}$ and $\dot{\mathbf{v}}$ tangential to the parametric curves at P. Thus, the tangent to C at P lies in the plane determined by the tangents at P to the parametric curves. If we move in parameter space by (du, dv), we will move by $d\mathbf{x}$ in position along the curve.

3.1 The First Fundamental Form

The differential arc is the magnitude of the change: $ds = |\dot{\mathbf{x}}|$. This differential arc ds is the distance between two points. We can can compute ds^2 as follows:

$$\begin{aligned} ds^2 &= (\mathbf{x}_u \cdot \mathbf{x}_u)du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v)dudv + (\mathbf{x}_v \cdot \mathbf{x}_v)dv^2 \\ &= Edu^2 + 2Fdudv + Gdv^2 \\ &= I \end{aligned}$$

This expression for ds^2 is known as the **first fundamental form** of the surface. Its constituent elements $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$ characterize the surface metrically. For instance, we can express the change in area as $dA = (EG - F^2)^{1/2}$ and the area of a surface patch is $\int\int (EG - F^2)^{1/2} dudv$.

The direction of the normal to the surface is $\mathbf{x}_u \times \mathbf{x}_u$. We normalize by its magnitude to find the unit normal:

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

We can also express the magnitude of the normal using E, F, and G:

$$\begin{aligned} (\mathbf{x}_u \times \mathbf{x}_v) \cdot (\mathbf{x}_u \times \mathbf{x}_v) &= (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2 \\ &= EG - F^2 \end{aligned}$$

resulting in

$$N = \frac{xu \times xv}{sqrt(EG - F^2)}.$$

3.2 The Second Fundamental Form

The first fundamental form allowed us to understand arc length and areas. We now move onward to curvature. Recall that we are discussing a curve on a surface. We now need to distinguish between the normal to the curve, which we have been writing as **n**, and the normal to the surface, which we shall call **N**. These two quantities are often different. An example is any circle on the sphere which is not a great circle: the normal to the circle is different from the normal to the sphere.

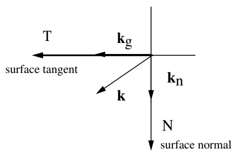

Figure 6: The total curvature can be decomposed into geodesic and normal components.

The tangent **t** is of course perpendicular to the normal and proportional to some vector which we will call **k**. We can decompose **k** into two components:

$$\begin{aligned} \mathbf{k} &= \mathbf{k}_n + \mathbf{k}_g \\ &= \kappa_n \mathbf{N} + \kappa_g \mathbf{T} \end{aligned}$$

These components are shown in Figure 6. The first component is proportional to the surface normal and is the **normal curvature** $\mathbf{k}_n$. The second is known as the **geodesic curvature** $\mathbf{k}_g$ or **tangential curvature**, and is intrinsic – it depends on the characteristics of the surface itself, not its embedding in three dimensional space. We focus for now on the normal curvature.

Note that the normal curvature of a curve on a surface is not the same as the normal curvature of a space curve. Here we discuss the former. The normal curvature vector $\mathbf{k}_n$ of all curves on a surface through a given point which share a tangent vector is the same. The normal curvature is the projection of **k** onto **N**: $\kappa_n = \mathbf{N} \cdot \mathbf{k} = \mathbf{N} \cdot \mathbf{t}'$. Since $\mathbf{N} \cdot \mathbf{t} = 0$, we can differentiate along the curve to find that $(\mathbf{N} \cdot \mathbf{t})' = \mathbf{N} \cdot \mathbf{t}' = -\mathbf{N}' \cdot \mathbf{t}$. Thus

$$\kappa_n = \mathbf{N} \cdot \mathbf{t}' = -\mathbf{N}' \cdot \mathbf{t} = -\frac{d\mathbf{N}}{ds} \cdot \frac{d\mathbf{x}}{ds} = \frac{-d\mathbf{N} \cdot d\mathbf{x}}{ds \cdot ds} = \frac{\mathbf{II}}{\mathbf{I}} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Recall the ds^2 is **I**, the first fundamental form. The quantity $-d\mathbf{N} \cdot d\mathbf{x}$ is the second fundamental form, also called **II** or the shape operator. It describes how the tangent (or normal) plane turns as you move around the surface. There are direct analogs to the elements E,F, and G found in first fundamental form, namely $e = -\mathbf{N}_u \cdot \mathbf{x}_u$, $2f = -(\mathbf{N}_u \cdot \mathbf{x}_v) + \mathbf{N}_v \cdot \mathbf{x}_u)$, and $g = -\mathbf{N}_v \cdot \mathbf{x}_v$. The properties of the shape operator are explored in great detail in classical differential geometry.

3.3 Meusnier's Theorem

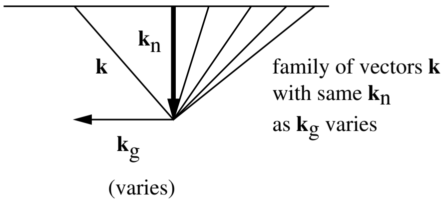

Figure 7: There is an entire family of curvature vectors which all share the same normal component.

Above we discussed the decomposition of **k** into the geodesic and normal components, as shown in Figure 6. Consider all the vectors **k** with the same normal component. These can be thought of as the intersection between the surface and a normal plane. Figure 7 shows that there is a family of such vectors which result from changing the geodesic component but leaving the normal component fixed. Thus all curves through the same point have the same normal curvature vector.

Actually, there are two different but interesting ways to move a plane about a normal vector: it can rock or roll. Figure 8 shows both cases. First, a family of planes which share a tangent can rock around the normal. All osculating circles to curves with the same tangent direction describe a sphere, sometimes known as Meusnier's sphere.

We can roll around the tangent vector to gain other degree of freedom In this case the osculating circles will change size as we rotate (unless the surface is a perfect sphere). The above discussion has been an informal statement of Meusnier's Theorem, which we will not prove.

The family of osculating circles that result from rocking around the tangent vector at a point has a well-defined minimum and maximum. The minimum and maximum circles are perpendicular to each other. The radii κ_1 and κ_2 of these two special circles are called the **principal radii of curvature** of the surface. The tangents are called the **lines of curvature**, and $t_1 \cdot t_2 = 0$. At every point on the surface we can compute these characteristic numbers which define the local curvature properties.

The radii of curvature are used to define the Gaussian and the mean curvature. The **Gaussian curvature** K is defined as $K = \kappa_1 \cdot \kappa_2$. The motivation for this definition is not immediately obvious, but we will discuss it in the next lecture.

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

is the appropriately-named **mean curvature**.

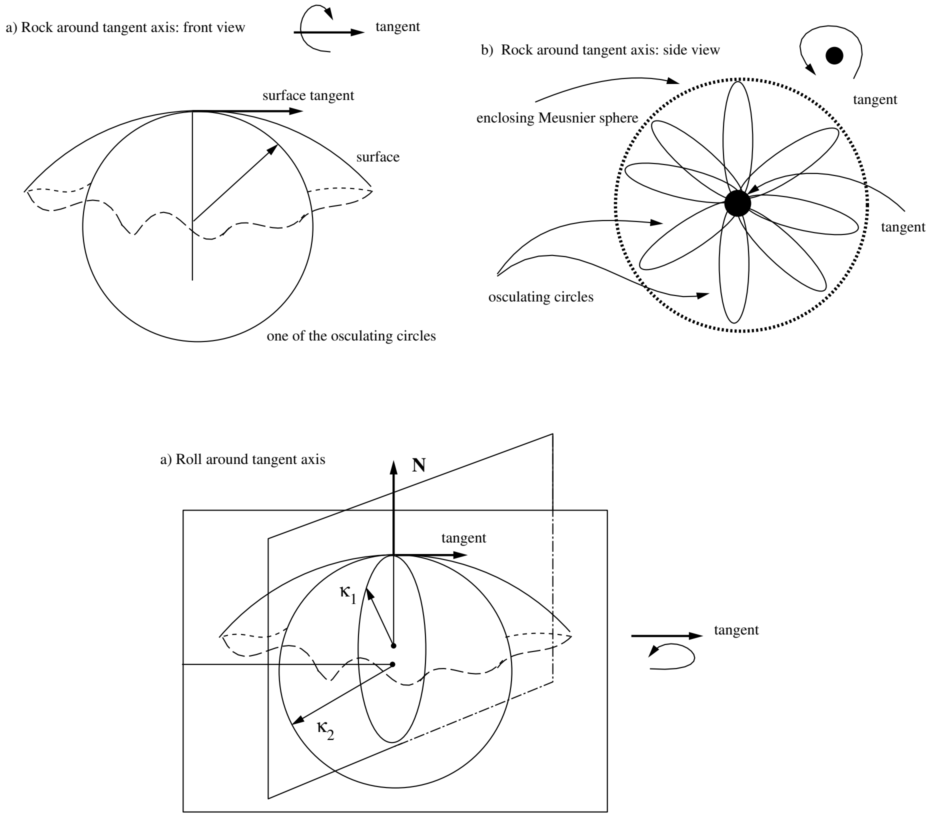

Figure 8: Meusnier's "Rock and Roll" Theorem. a) Rocking around the tangent vector on the normal plane: there are a whole family of circles which share the same tangent on the surface. b) A side view of the rocking, looking down onto the tangent axis, showing the rotating osculating circles which form Meusnier's sphere. c) Rolling around the tangent vector on the normal plane. The size of the osculating circle changes, and the maximum and minimum sizes appear in perpendicular directions. The radii of curvature κ_1 and κ_2 of these extremal circles are shown.

3.4 Ellipsoid

The curvature vectors on a surface are uniquely determined. Figure 9 shows the lines of curvature on an ellipsoid. In this case κ_1 and κ_2 are both positive: that is, they are both on the same side of the tangent plane. The lines of curvature form a net: they always intersect at right angles, and thus form a nice natural parametrization for the surface. These lines can be computed by starting at some point and integrating. The dot in the figure is called an **umbilic point** and is the place where minimimum and maximum curvatures are the same. At this point the local neighborhood of the surface can be approximated by a perfect sphere.

We can also define a series of concentric ellipsoids. Consider the 3D net we could define by connecting up the lines of curvature on these offset ellipsoids. These new

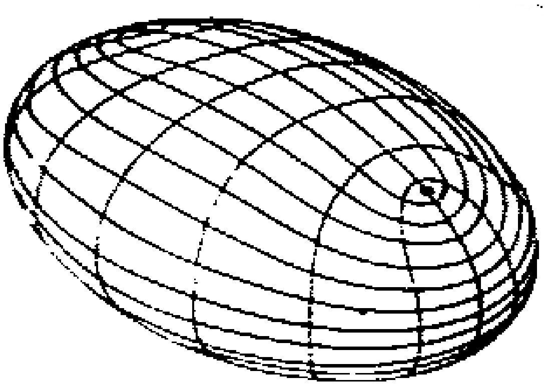

Figure 9: Lines of curvature meet at right angles and form a net on an ellipsoid. The minimimum and maximum curvatures are the same at the dot, which is an umbilic point. Reproduced from [3], Figure 2.22

surfaces would all meet at right angles and thus form a nice 3D curvilinear coordinate system.

3.5 Cuboid

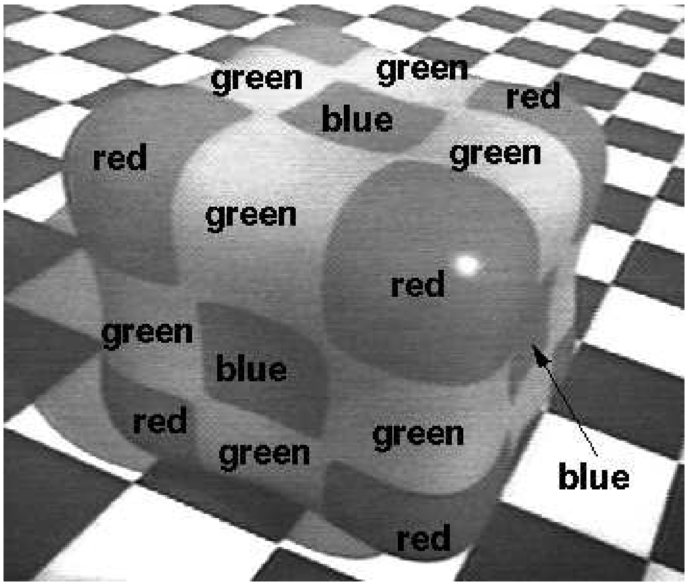

Figure 10: A cuboid surface colored by Gaussian curvature. Reproduced from [2], Plate 1

Figure 10 shows a cuboid surface colored by Gaussian curvature. The figure is annotated with a color legend since these notes are distributed in black and white. The red parts at the "corners" of the cuboid are elliptic – they are locally convex and both the

Gaussian curvature K and the principal radii κ_1 and κ_2 are positive. The blue patches in the center of the faces are concave: the Gauss curvature is still positive although the radii κ_1 and κ_2 are both negative. The center of curvature is on the opposite side of the tangent plane. The green parts of the surface along the edges have negative Gauss curvature. They are saddle points where the principle radii of curvature have different signs. Two colors meet along **parabolic lines**. These lines of inflection separate the positive and negative Gaussian curvature regions.

3.6 The Apollo of Belvedere

Figure 11: The Apollo of Belvedere with parabolic lines drawn in by Felix Klein. Reproduced from [1], Figure 440.

Figure 11 shows a reproduction of the Apollo of Belvedere. Parabolic lines were hand-drawn on its surface at the behest of mathematician Felix Klein. He hoped that decomposing a complex shape along parabolic lines would illustrate its fundamental aesthetics. Supposedly he abandoned this theory after seeing the resulting picture. Harvard mathematics professor David Mumford has looked at the problem more recently. Finding the true parabolic lines is tricky and is rather sensitive to noise. Stanford graphics professor Marc Levoy will be scanning the real Apollo during his Digital Michelangelo Project in Italy next year, at which time the parabolic lines could be computed directly to satisfy the curiosity of any aspiring aestheticians among us.

All theories of aesthetics aside, parabolic lines have many nice properties. Some of these pertain to caustics, which we will hear about in the next lecture.

References

[1] Jan J. Koenderink. *Solid Shape.* MIT Press, 1990.

[2] Don Mitchell and Pat Hanrahan. Illumination from curved reflectors. In *Computer Graphics (Proc. SIGGRAPH 1992)*, July 1992.

[3] Dirk J. Struik. *Lectures on Classical Differential Geometry.* Dover, 2nd edition, 1950.