CS448: Topics in Computer Graphics Lecture #15
Mathematical Models for Computer Graphics
Stanford University Tuesday, 1 December 1997

Classical Subdivision Surfaces II

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In the previous lecture, we discussed the Loop subdivision scheme which works for triangular meshes. In this lecture we will look at several other subdivision schemes.

1 Background

Each subdivision scheme can be thought of as two rules: the topological refinement rule that describes how the refined mesh is created from the original mesh, and the subdivision rule, that is used to compute the locations of the vertices of the new mesh. The first rule, which we will call simply the *refinement rule*, describes how many new vertices are added to the mesh and which vertices in the new mesh are connected by edges. For example, the Loop scheme replaces a triangle at subdivision level j with 4 triangles at level $j+1$. The new mesh at subdivision level $j+1$ is created by splitting each edge and connecting the inserted vertices as shown in Figure 1.

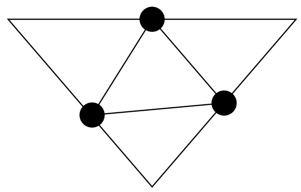

Figure 1: Connecting the 3 new vertices creates 4 new level $j+1$ triangles.

Two of the schemes that we consider in this lecture are defined for quadrilateral rather then triangular meshes. For such meshes we have two main options for refinement rules.

The first refinement rule, used by the *Doo-Sabin* subdivision scheme, first builds a level $j+1$ quadrilateral in the center of each existing level j face quadrilateral. The new mesh is obtained by connecting each vertex of the level $j+1$ quadrilateral with its three neighbors to produce 3 more level $j+1$ quadrilaterals. This refinement rule is illustrated in Figure 2.

The second refinement rule, which is used by the *Catmull-Clark* subdivision scheme subdivides a level j quadrilateral into 4 level $j+1$ quadrilaterals. New vertices are created by splitting each edge into two and creating a new vertex for each face. This method is illustrated in Figure 3.

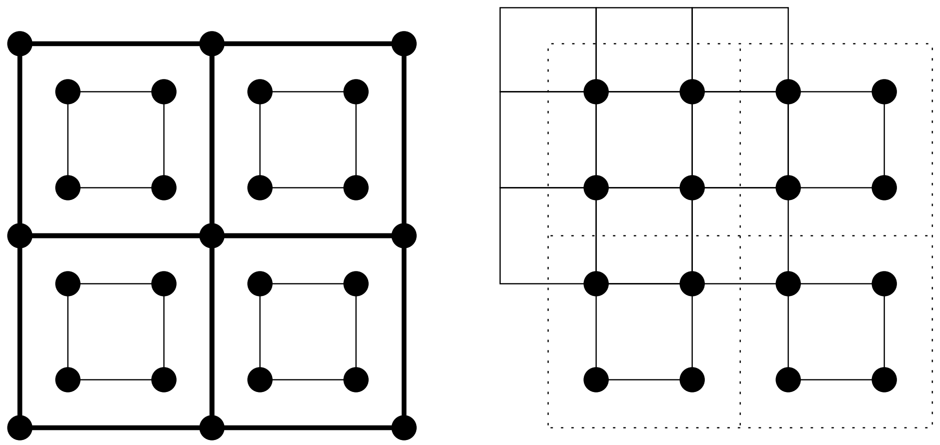

Figure 2: Refinement rule used by Doo-Sabin subdivision scheme. Step 1: New vertices are added to create level *j+1* quadrilaterals in the center of level j quadrilaterals. Step 2: The vertices of the center face quadrilaterals are connected to their neighbors. Each level j quadrilateral is covered by parts of 9 level *j+1* quadrilaterals, but there are only 4 level *j+1* quadrilaterals created for each level j quadrilateral. The vertices of the level j quadrilaterals are discarded. This refinement rule works for even degree tensor-product splines.

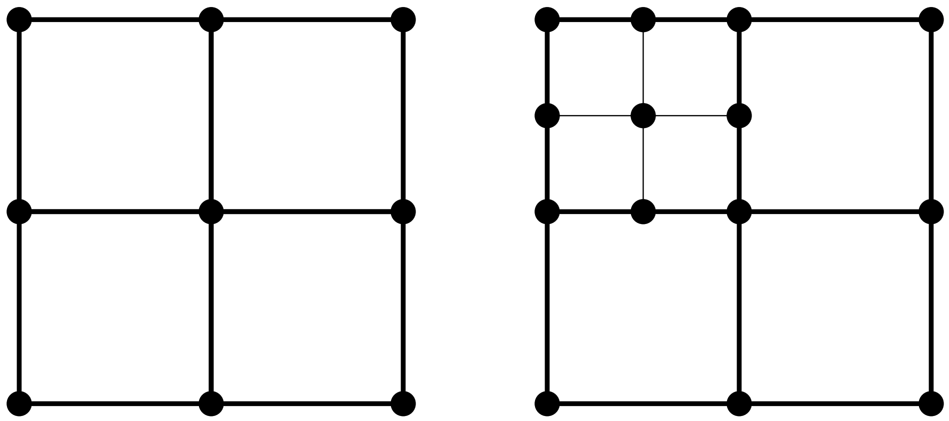

Figure 3: Refinement rule used by Catmull-Clark subdivision scheme. New vertices are added on each edge and in the center. When connected, 4 new level *j+1* quadrilaterals are produced from the single level j quadrilateral. This refinement rule works for odd degree tensor product splines.

2 Doo-Sabin

2.1 Quadrilaterals

The Doo-Sabin subdivision scheme is based on quadratic splines. Recall that a generating function for one-dimensional quadratic splines is $(z+1)^3/4$. For a tensor product spline of total degree 4 the generating function is

$$f(z_1, z_2) = \frac{(1+z_1)^3(1+z_2)^3}{2^2 \cdot 2^2}$$

If we multiply this out we end up with:

$$f(z_1, z_2) = \frac{(1+3z_1+3z_1^2+z_1^3)(1+3z_2+3z_2^2+z_2^3)}{16}$$

The coefficients of the monomials of $f(z_1, z_2)$ are the coefficients of the subdivision scheme.

We can arrange the coefficients into a table, with the coefficient of $z_1^i z_2^j$ in position (i, j):

$$\frac{1}{16} \cdot \begin{pmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

For quartic (that is, of total degree 4) tensor-product splines we have 4 rules for determining the positions on the vertices on the next subdivision level. They are the Even-Even, Even-Odd, Odd-Even and Odd-Odd rules. Each rule uses 4 coefficients from the above table. The name of each rule determines which 2 rows (Even or Odd) and which two columns (Even or Odd) of the table are selected. The Even-Even rule, for example selects from the even column and even row giving the coefficients $\frac{1}{16}$, $\frac{3}{16}$, $\frac{3}{16}$, and $\frac{9}{16}$. Geometrically, this results in a point EE that is $\frac{1}{16}C_0 + \frac{3}{16}C_1 + \frac{3}{16}C_2 + \frac{9}{16}C_3$ as shown in Figure 4.

2.2 Doo-Sabin subdivision: N-gons

Doo-Sabin subdivision extends the rules for quartic splines described above to general polygonal meshes. For each N-gon at level j, we create a level *j+1* N-gon. This is similar to dealing with extraordinary points in the Loop scheme, but now we cannot use symmetry to compute all coefficients once one of them is known. The rules for the vertices of the new N-gon can be derived using the eigenstructure of the subdivision matrix. Suppose we are computing a new vertex of the N-gon on level $j+1$. This vertex is a linear combination of the vertices of the old N-gon. Suppose these vertices are numbed from 0 to $N-1$ starting with the vertex nearest to the vertex on level $j+1$ that we are computing. Then the coefficients are $\alpha_0 = \frac{1}{4} + \frac{5}{4N}$, $\alpha_k = \frac{3+2cos(\frac{2\pi k}{N})}{4N}$, $k = 1 \ldots N-1$ where k is the vertexes number.

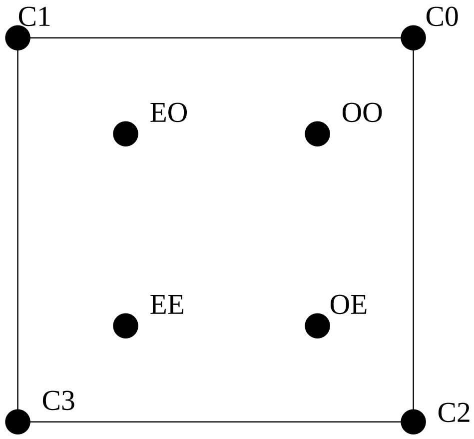

Figure 4: The level *j+1* point determined by the Even-Even rule is at $\frac{1}{16}C_0 + \frac{3}{16}C_1 + \frac{3}{16}C_2 + \frac{9}{16}C_3$. The other points are determined in a similar manner. All rules are the same up to a renumbering of the points.

3 Catmull-Clark

3.1 Quadrilateral Rules

The Catmull-Clark subdivision scheme is based on cubic splines.

Its generating function is:

$$f(z_1, z_2) = \frac{(1+z_1)^4 \cdot (1+z_2)^4}{64}$$

In this case, there are three different rules called the *vertex rule*, the *edge rule* and the *face rule* for computing point positions. These rules also correspond to various choices of columns and rows of the table of coefficients of the generating function: Even-Even, Odd-Even etc. The vertex rule computes a new position of an existing vertex using the vertex itself and 8 immediate neighbors. The edge rule computes the position for a new vertex inserted on an edge between two existing vertices. It uses the closest 6 existing vertices. The simplest rule, the face rule, creates a new point in the middle of a quadrilateral face. All rules are shown in Figure 5.

3.2 Extraordinary Vertices

For extraordinary vertices, Catmull-Clark scheme uses modified rule similar to Loop's scheme. Since there are quadrilaterals meeting at the extraordinary point instead of triangles, we have two sets of vertices that should have equal coefficients by symmetry. We call them *inner* and *outer* vertices (Figure 6). This leaves us two degrees of freedom.

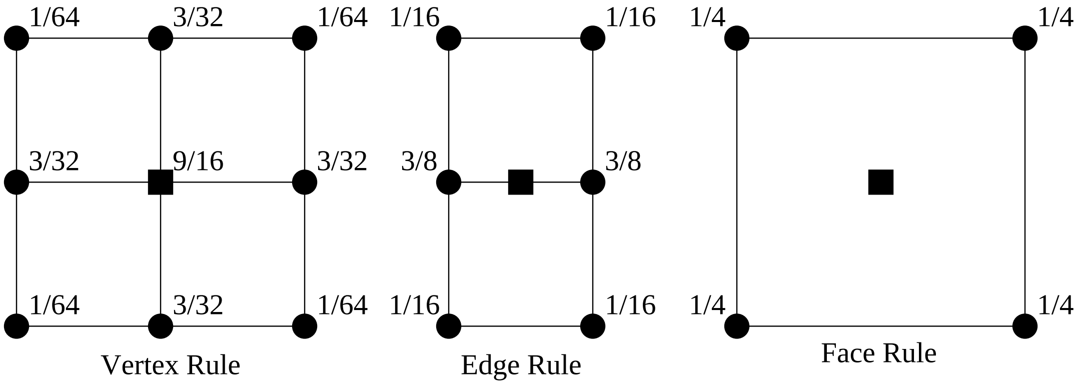

Figure 5: Each black circle represents a vertex at level j; we compute the position of the vertex at level $j+1$ marked by by the black square. Note that for the vertex rule, the control vertex with weight $\frac{9}{16}$ and the new vertex aren't necessarily aligned as they are in the figure.

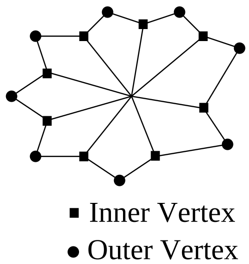

Figure 6: Inner and outer vertices surrounding an extraordinary point in a quadrilateral mesh.

If the coefficients of inner vertices are $\frac{\beta}{N}$ and the coefficients of all outer vertices are be $\frac{\gamma}{N}$, where N is number of inner vertices. The coefficient of the central vertex is $1-\beta-\gamma$ by affine invariance. The values for β and γ which Catmull and Clark came up with are

$$\begin{aligned} \beta &= \frac{3}{2N} \\ \gamma &= \frac{1}{4N} \end{aligned}$$

Note that for $N=4$, we do get the coefficients for the spline vertex rule. Also, note that the weight on an inner or outer point is proportional to $\frac{1}{N^2}$ - we do divide by N twice. These coefficients are by no means unique: there is a range of β and γ that leads to smooth surfaces.

3.3 Arbitrary Meshes

We have defined Catmull-Clark scheme on quadrilaterals; it can be extended to handle arbitrary polygonal meshes. Observe that if we do one step of refinement, splitting each edge into two and inserting a new vertex for each face (Figure 7), we get a mesh which has only quadrilateral faces. Special rules described in Catmull and Clark's paper are used for this first step. On all other steps of subdivision standard rules described above can be applied.

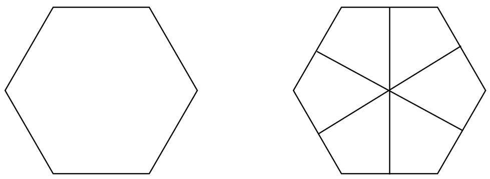

Figure 7: Splitting a hexagon into quadrilaterals.

4 Butterfly Rule : Interpolation on a Triangular Grid

The Butterfly subdivision scheme is similar to the Loop scheme, but it creates an interpolated surface which contains the vertices of the original triangular mesh. The Loop scheme had two rules, one for changing positions of old vertices and one for computing positions of new vertices. Since the Butterfly scheme is interpolating, we don't need a rule for changing the positions of old vertices. If we want the scheme to generate smooth surfaces, we do need wider support for the rule that adds a vertex. To add a new vertex we use the coefficients shown in Figure 8.

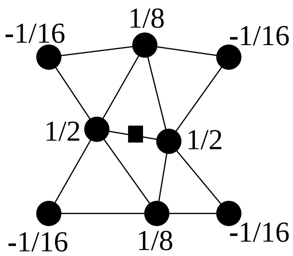

Figure 8: Each black circle is a control point with the corresponding weight. The black square is the new vertex.

This scheme is based on the one-dimensional four-point interpolating scheme with coefficients $\frac{-1}{16}, \frac{9}{16}, \frac{9}{16}, \frac{-1}{16}$. This subdivision scheme generates smooth interpolating surfaces only for regular meshes. Smoothness, however, is not guaranteed at extraordinary points. A modification of this scheme with special coefficients for computing new vertices adjacent to the extraordinary vertices generates surfaces that are smooth everywhere for almost all configurations of control vertices. The coefficients of the scheme are given by

$$s_j = \frac{1}{n}\left(\frac{1}{4} + \cos\frac{2\pi}{n} + \frac{1}{2}\cos\frac{2\pi}{n}\right),\ j = 0\ldots n-1$$

if the extraordinary vertex has degree $n \geq 5$. The coefficients are $5/12, -1/12, -1/12$ for $n = 3$ and $3/8, 0, -1/8, 0$ for $n = 4$.

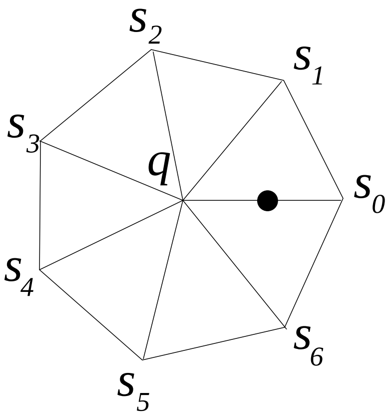

Figure 9: Rule for inserting a new vertex next to an extraordinary vertex for the Modified Butterfly scheme.

5 Creases

The Loop scheme can be modified in such a way that a path of control vertices on the coarsest level can be "marked" meaning the regular Loop scheme rule is replaced with an alternative scheme. To create a crease, the normal weights of $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ are replaced with weights $0, \frac{1}{2}, \frac{1}{2}, 0$ for adding new points on the crease. For changing the position of old points on the crease we weight the current position with $\frac{3}{4}$ and its two neighbors on the crease edge with $\frac{1}{8}$. This rule replaces the normal Loop scheme rule for changing the position of old vertices. The effect is that any vertex on the path won't be influenced by the rest of the surface, it will only be influenced by points already on the path. The downside of this approach is that the surface may behave badly as it approaches this independently defined curve.