**Analysis of Subdivision Curves**

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Lecturer: Denis Zorin
Scribe: Hoofar Razavi

# 1 Introduction

As we saw in the last lecture, the basis functions used for generating a B-spline of degree $n$ is derived from the convolving the box function with itself $n$ times:

$$N(t) = \bigotimes_{i=0}^{n} U(t)$$

where

$$U(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\otimes$ denotes convolution. These basis functions all have local support and are piecewise polynomial with degree $n$. However, more importantly, they also satisfy the dilation equation, which we first encountered in the context of wavelets. The dilation equation for the B-spline basis is simply:

$$N(t) = \sum_k s_k N(2t - k) \tag{1}$$

From before we also know that if a set of basis functions satisfy the dilation equation multiresolution techniques such as successive refinement can be applied to functions in the spaces these basis functions span. We set out to find $s_k$ so that we can apply these techniques to the representation of our curves. If you are familiar with DeBoor's knot insertion algorithm, the above formula can be interpreted as a special case of DeBoor's formula.

# 2 The Derivation

Recall the following properties of convolution for functions $f(t)$, $g(t)$, and $h(t)$, given that $m(t) = f(t) \otimes g(t)$:

$$\begin{aligned} f(t) \otimes (g(t) + h(t)) &= f(t) \otimes g(t) + f(t) \otimes h(t) && \text{linearity} \\ f(t-i) \otimes g(t-k) &= m(t-i-k) && \text{time shift} \\ f(2t) \otimes g(2t) &= \tfrac{1}{2} m(2t) && \text{time scaling} \end{aligned}$$

Also, recall from last time that the box function can be written in terms of its own dilates

$$U(t) = U(2t) + U(2t-1)$$

Now, looking at the first degree basis and using the above observation we have

$$\begin{aligned} N(t) &= U(t) \otimes U(t) && \text{first degree basis function} \\ U(t) &= U(2t) + U(2t-1) \\ N(t) &= (U(2t) + U(2t-1)) \otimes (U(2t) + U(2t-1)) \end{aligned}$$

Rewriting the above using the convolution properties presented we get

$$\begin{aligned} N(t) &= U(2t) \bigotimes U(2t) + U(2t) \bigotimes U(2t-1) + U(2t) \bigotimes U(2t-1) + U(2t-1) \bigotimes U(2t-1) \\ &= \frac{1}{2}N(2t) + \frac{1}{2}N(2t-1) + \frac{1}{2}N(2t-1) + \frac{1}{2}N(2t-1-1) \\ &= \frac{1}{2}N(2t) + N(2t-1) + \frac{1}{2}N(2t-2) \end{aligned}$$

Comparing the above derivation to Equation 1 we arrive at the coefficients $s_0 = \frac{1}{2}$, $s_1 = 1$, and $s_2 = \frac{1}{2}$. We have managed to rewrite the basis function for a first degree B-spline in terms of *its* own dilates as well. The above derivation generalizes to B-spline basis functions of higher degrees. Now we compute the coefficients $s_k$ for degree $n-1$ splines splines

$$\begin{aligned} N_{n-1}(t) &= \bigotimes_{i=0}^{n} U(t) \\ &= \bigotimes_{i=0}^{n} (U(2t) + U(2t-1)) \end{aligned}$$

With the derivation of the coefficients of the first degree spline, we make the following observation. The coefficients of the translates of the dilate of N(t) in equation 1 are simply the elements of the matrix $S$. Recall that the binomial expansion derives the coefficients for each term of the exponentiation:

$$(x+y)^n = \sum_{i=0}^{n} a_i x^{n-i} y^i$$

where

$$a_i = \begin{pmatrix} n \\ i \end{pmatrix}$$

Similarly, because the convolution operator is linear, just like multiplication, we can derive our coefficients simply through the binomial expansion by rewriting the above equation.

$$\begin{aligned} N(t) &= \underbrace{(U(2t)+U(2t-1))\bigotimes(U(2t)+U(2t-1))\bigotimes(U(2t)+U(2t-1))}_{n}\dots \\ &= \sum_{i=0}^{n} s_i \underbrace{U(2t)\bigotimes U(2t)\bigotimes\dots}_{i}\underbrace{U(2t+1)\bigotimes U(2t+1)\bigotimes\dots}_{n-i} \end{aligned}$$

From the above it is easily seen that the $s_k$ coefficients are none other than the coefficients of the binomial expansion for $(x+y)^n$ divided by $2^n$.

# 3 Representing Curves

Consider a point in the plane

$$\begin{bmatrix} c_x^i \\ c_y^i \end{bmatrix}$$

to be the $i^{th}$ control point of a curve we would like to represent. The curve, $\gamma(t)$, is thus represented as

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \sum_i \begin{bmatrix} c_x^i \\ c_y^i \end{bmatrix} N(t-i)$$

As we saw in the last lecture, each of the translates of $N(t)$ has a maximum value at some point during the interval it is nonzero. Writing our curve in this manner ensures that each control point influences the curve as well as guaranteeing that such influence is limited, due to the local support of translates of $N(t)$.

We can rewrite the above more compactly as

$$\gamma(t) = \sum_i c_i N(t-i)$$

Now consider $\mathbf{c}$, the vector of control points of a given curve:

$$\mathbf{c} = \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix}$$

and the vector $\mathbf{N}(t)$, which has as its elements the translates of the function N, the desired basis function, defined above:

$$\mathbf{N} = \left[ \begin{array}{ccccccc} \ldots & N(t+2) & N(t+1) & N(t) & N(t-1) & N(t-2) & \ldots \end{array} \right]$$

In this notation we can denote our curve as $\mathbf{Nc}$. Since we know that each of the elements of $\mathbf{N}$ can be represented in terms of its dilates we rewrite the whole vector as such

$$\mathbf{N}(2t) = \left[ \begin{array}{ccccccc} \ldots & N(2t+2) & N(2t+1) & N(2t) & N(2t-1) & N(2t-2) & \ldots \end{array} \right]$$

Where we relate the two vectors with the matrix $S$ as follows:

$$\mathbf{N}(t) = \mathbf{N}(2t)S$$

Rewriting $\gamma(t)$ based on the above representation yields:

$$\gamma(t) = \mathbf{N}(2t)S\mathbf{c}$$

We can think of $\mathbf{N}(2t)$ as our new basis and $S\mathbf{c}$ as our new control points. Since each successive rewriting of the function $\mathbf{N}$ uses basis functions that are twice as narrow as the functions on the previous step, they are shifted by one half of the shift on the previous step. The new control points correspond to values of $t$ which are twice as dense as on the previous level. As we will see below, the new control points are simply linear combination of the old points since they are related through the matrix $S$. The elements of $S$ are related to the coefficients $s_k$ in Equation 1 by

$$S_{2i+k,i} = s_k$$

Representing our curve, using the above, at each degree gives us a set of successive refinements of the curve:

$$\begin{array}{rcllc} \gamma(t) & = & \mathbf{N}(t)\mathbf{c}^0 & & \text{Zeroth degree} \\ & = & \mathbf{N}(2t)\mathbf{c}^\mathbf{1} & = \mathbf{N}(t)S\mathbf{c}^\mathbf{0} & \text{First degree} \\ & & \vdots & & \\ & = & \mathbf{N}(2^j t)\mathbf{c}^\mathbf{j} & = \mathbf{N}(t)S^j\mathbf{c}^\mathbf{0} & J^{th} \text{ degree} \end{array}$$

from which we can see the relationship between control points at different levels:

$$\mathbf{c}^\mathbf{j+1} = S\mathbf{c}^\mathbf{j}$$

where $S$ is our infinite subdivision matrix. Looking more closely at one component, $i$, of our control points we see that:

$$c_i^{j+1} = \sum_l S_{i,l} c_l^j$$

To find out exactly which $s_k$ is affecting which term, we can divide the above into odd and even entries. For the odd entries we have:

$$c_{2i+1}^{j+1} = sum_l S_{2i+1,l} c_l^j = sum_l s_{2(i-l)+1} c_l^j$$

and for the even entries we have

$$c_{2i}^{j+1} = sum_l S_{2i,l} c_l^j = sum_l s_{2(i-l)} c_l^j$$

From which we essentially get two different subdivision rules one for the new even points of the curve and one for the new odd points. As examples of all of the above, let us consider a couple of concrete cases. For piecewise *linear* subdivision, the basis functions are hat functions. The odd coefficients are $\frac{1}{2}$ and $\frac{1}{2}$, and 1 for the even point. For second degree splines, the odd coefficients turn out to be $\frac{1}{4}$ and $\frac{3}{4}$, and $\frac{3}{4}$ and $\frac{1}{4}$ for the even ones.

# 4 Discrete Convolution

The coefficients $s_k$ can also be derived from another perspective, namely discrete convolution. More explicitly, using generating functions for a sequence $x_k$ we can consider a polynomial

$$X(z) = \sum_k x_k z^k$$

where $X(z)$ is the $z$-transform of the sequence $x_k$; for the case of two coefficients we have

$$(a \bigotimes b)_k = \sum_n a_{k-n} b_n$$

Suppose we have two functions that satisfy the dilation equation:

$$\begin{aligned} f(t) &= \sum_k h_0(k).f(2t-k) \\ g(t) &= \sum_k h_1(k).g(2t-k) \end{aligned}$$

Then $f \otimes g$ also satisfies the dilation equation with coefficients $h'(k) = 1/2 h_0 x h_1$ where

$$h'(k) = \frac{1}{2} \sum_i h_0(k-i) h_1(i)$$

Since $U(t)$ satisfies the dilation equation $U(t) = U(2t) + U(2t-1)$, $N(t) = X_{i=0}^{n-1} U(t)$ satisfies a dilation equation $N(t) = \sum_k s_k N(2t-k)$ with

$$s_k = (1/2)[...0110...]x(1/2)[...0110...]x...x(1/2)[...0110...] \quad (n \text{ times}).$$

The z-transform of [... 0 1 1 0 ...] is 1+z.

We obtain the following generating function for splines:

$$S(z) = \frac{(1+z)^n}{2^{n-1}}$$

Again, the values $s_k$ become apparent: they are simply the coefficients of each of the terms in the generating function above.

# 5 Interpolation vs. Approximation

Splines are not interpolating, that is, they do not pass through their control points. We can generate interpolating curves by choosing the even rows of the $S$ matrix to be $\ldots 0, 0, 1, 0, \ldots$, so that the control points, once added are never moved. Then we choose the coefficients for the odd points in such a way that the generated curves are smooth. One such choice is the four point scheme, first introduced by Dyn, Levin, and Gregory), which has generating function

$$S(z) = \frac{1}{16}(-z^{-3} + 4z^{-2} - z^{-1})(1+z)^4$$

Recall that convolving with a box increases smoothness and convolution with $U(t)$ means multiplication of the generating function by $(1+z)/2$. We have four factors $(1+z)/2$ in the generating function above, but three of them are required just to get subdivision to converge. One extra term $(1+z)/2$ ensures that the curve is smooth.

In general, the difficult part is to find a set of coefficients for which subdivision converges. There is no general method to achieve this; once we have a convergent subdivision scheme, we can obtained desired smoothness by convolving with the box.

# 6 Subdivision vs. Splines

The subdivision coefficients $s_k$'s need not be fixed as we presented them so far. We can vary them, both between levels of subdivision and within one level, between points to obtain different curves. In this regard, splines are just a special case of the more general class of curves, subdivision curves.

There is no sufficiently strong reason for using subdivision in one dimension (in fact none of the commercial line drawing packages do) but the argument becomes much more compelling in higher dimensions.