Biorthogonal Wavelets

Lecture #4: Thursday, 9 October 1997
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In the previous lecture we dropped symmetry in favor of orthogonality, and thus arrived at the Daubechies wavelets. This time around we relinquish orthogonality and settle for the next best thing, *biorthogonality*. Without orthogonality, the construction of our filters becomes subject to far fewer constraints; as a result, the process of meeting the other desired criteria is more involved.

1 Giving Up on Orthogonality

Recall the set of properties we identified as desirable for a wavelet filter:

1. Finite Impulse Response.
2. Orthogonality.
3. Good approximation.
4. Symmetry.
5. Regularity (smoothness).

Unfortunately, enforcing FIR, orthogonality, and symmetry together yields nothing better than the Haar case. To see this, consider a symmetric signal of length, say, 6:

$$\begin{array}{ccccccc} & c(0) & c(1) & c(2) & c(3) & c(4) & c(5) \\ = & c(0) & c(1) & c(2) & c(2) & c(1) & c(0) \end{array}$$

Now consider various even shifts of this signal:

$$\begin{array}{cccccccccc} c(0) & c(1) & c(2) & c(2) & c(1) & c(0) & & & & \\ 0 & 0 & c(0) & c(1) & c(2) & c(2) & c(1) & c(0) & & \\ 0 & 0 & 0 & 0 & c(0) & c(1) & c(2) & c(2) & c(1) & \cdots \end{array}$$

By hypothesis, $c(0) \neq 0$ (since the signal has length 6). Because of the orthogonality condition

$$\sum_n c(n)c(n-2k) = \delta(k)$$

we must have $c(2)c(0) = 0$ hence $c(2) = 0$, and $c(1)c(0) = 0$ hence $c(1) = 0$. Every coefficient except $c(0)$ must be 0, so all finite, symmetric, orthogonal filters have the form

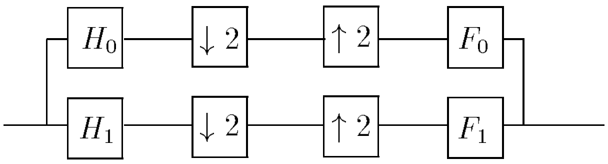
$$c(0) \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad c(0)\ .$$

Of those, the only reasonable low-pass filter is the one with length 2, i.e., the Haar filter.

2 Perfect Reconstruction from General Filter Banks

In Lecture #1 we saw that if we use the Haar filter pair in our analysis bank, all we need for perfect reconstruction of the input signal is another copy of the Haar filter pair, with the sign of H_1 flipped, in our synthesis bank. But what are the conditions for perfect reconstruction in the general case?

Here is the general structure we will consider:



analysis bank synthesis bank

The analysis bank can be represented as a single matrix $\left[\frac{L}{B}\right]$, where L corresponds to H_0 followed by downsampling and B corresponds to H_1 followed by downsampling. In the case of an orthogonal filter bank, $\left[\frac{L}{B}\right]$ is orthogonal. It is also possible to use a non-orthogonal $\left[\frac{L}{B}\right]$, but for perfect signal reconstruction the synthesis bank must be of the form $\left[\ \tilde{L} \ \middle|\ \tilde{B}\ \right]$ such that

$$\left[\ \tilde{L} \ \middle|\ \tilde{B}\ \right] \cdot \left[\frac{L}{B}\right] = \tilde{L} \cdot L + \tilde{B} \cdot B = I$$

as well as

$$\left[\frac{L}{B}\right] \cdot \left[\ \tilde{L} \ \middle|\ \tilde{B}\ \right] = \left[\begin{array}{c|c} L \cdot \tilde{L} & L \cdot \tilde{B} \\ \hline B \cdot \tilde{L} & B \cdot \tilde{B} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right] \tag{1}$$

(where each I represents an identity matrix and each 0 a zero matrix). Equation (1) gives us constraints in terms of coefficients:

$$L \cdot \tilde{L} = I \iff \sum_n h_0(k) f_0(k - 2n) = \delta(n)$$

$$L \cdot \tilde{B} = 0 \iff \sum_n h_0(k) f_1(k - 2n) = 0 \tag{2}$$

$$B \cdot \tilde{L} = 0 \iff \sum_n h_1(k) f_0(k - 2n) = 0 \tag{3}$$

$$B \cdot \tilde{B} = I \iff \sum_n h_1(k) f_1(k - 2n) = \delta(n)$$

3 Alternating Flip

Conditions (2) and (3) above denote the biorthogonal nature of the system: L is orthogonal to $\tilde{B}$ and B is orthogonal to $\tilde{L}$. We can ensure that some of the biorthogonality conditions are satisfied requiring that the coefficients of the mutually orthogonal filters be the same but in reverse order and with every other sign flipped:

$$f_0(k) = (-1)^k h_1(N-k)$$

$$f_1(k) = -(-1)^k h_0(N-k)$$

(N is the order of the filters.)

It also turns out that, if we use alternating flip, $L \cdot \tilde{L} = I$ is equivalent to $B \cdot \tilde{B} = I$; so we are now left with only one condition to worry about: how to choose f_0 and h_0.

We no longer have a single dilation equation as in the orthogonal case, but rather a pair of dilation equations:

$$\phi(t) = 2\sum_k h_0(k)\phi(2t-k)$$

$$\tilde{\phi}(t) = 2\sum_k f_0(k)\tilde{\phi}(2t-k)$$

What does this mean in the continuous domain? There is a nice interpretation in terms of bases; but first we should introduce the concept of *multiresolution analysis*.

4 Multiresolution Analysis

All wavelets exhibit a multiresolution structure in the following sense: they are made up of nested sets of increasingly refined and powerful bases. Given a scaling function ϕ and its associated wavelet function w, we can form the space

$$V_0 = \text{span}\langle\phi(t-k)\rangle = \text{span}\langle\phi(2^0 t-k)\rangle$$

spanned by the translates of ϕ, and the space

$$W_0 = \text{span}\langle w(t-k)\rangle = \text{span}\langle w(2^0 t-k)\rangle$$

spanned by the translates of w. By scaling ϕ we can also build

$$V_1 = \text{span}\langle w(2t-k)\rangle = \text{span}\langle w(2^1 t-k)\rangle,$$

which (provided that w is the wavelet derived from ϕ) will be exactly the sum of V_0 and W_0:

$$V_1 = V_0 + W_0$$

What this means is that any function in V_1 can be represented as the sum of a function in V_0 and a function in W_0. Further scaling of ϕ and w by powers of 2 allows us to define an infinite sequence of spaces V_j and W_j:

$$V_j = \text{span}\langle \phi(2^j t - k) \rangle$$

$$W_j = \text{span}\langle w(2^j t - k) \rangle$$

At every level we have $V_j = V_{j-1} + W_{j-1}$, which implies the infinite nesting of the bases:

$$V_0 \subset V_1 \subset V_2 \subset \ldots$$

Note: All of the above applies in both orthogonal and biorthogonal cases; but if the V_j's and W_j's are mutually orthogonal, i.e.,

$$V_j \perp W_j \text{ for all } j$$

or (by definition)

$$\langle f_{V_j} \cdot f_{W_j} \rangle = 0 \text{ for all functions } f_{V_j} \in V_j \text{ and } f_{W_j} \in W_j,$$

then we have *orthogonal* multiresolution and we write $V_j = V_{j-1} \oplus W_{j-1}$ instead of $V_j = V_{j-1} + W_{j-1}$.

In summary, here are the defining properties of a multiresolution set $\{V_j\}$ of functional spaces:

1. $V_j \subset V_{j+1}$ and $\bigcup_j V_j = L^2$
2. $f(t) \in V_j \iff f(2t) \in V_{j+1}$
3. $f(t) \in V_j \iff f(t-k) \in V_j$

5 Dual Bases

We start with finite-dimensional spaces. Let y be a vector such that $y = \sum a_i y_i$ for some set of coefficients a_i. If the y_i's form an orthonormal basis, finding the coefficients a_i is particularly straightforward: each a_i is equal to the inner product $\langle y_i, y \rangle$.

If the $\{y_i\}$ basis is not orthogonal, however, we lose the luxury of such a simple computation. But it turns out that it is possbile to find a set of vectors $\{\tilde{y}_i\}$ such that

$$a_i = \langle \tilde{y}_i, y \rangle. \tag{4}$$

Since $y = \sum a_i y_i$, we have

$$\langle \tilde{y}_i, y \rangle = \langle y, \tilde{y}_i \rangle = \sum_j a_i \langle y_i, \tilde{y}_j \rangle$$

and the $\tilde{y}_i$ vectors are therefore determined by the set of conditions

$$\langle y_i, \tilde{y}_j \rangle = \delta(i-j) \tag{5}$$

or, in matrix notation,

$$Y \cdot \tilde{Y} = I.$$

This property makes $\{y_i\}$ and $\{\tilde{y}_i\}$ *dual bases.* Two bases are said to be duals of each other if there exists a one-to-one correspondence between them such that the inner product of corresponding basis elements is 1 while the inner product of non-corresponding elements is 0.

The same concept applies to functional spaces. If a filter bank provides perfect reconstruction of the input signal, the bases generated by the synthesis functions $\tilde{\phi}$ and $\tilde{w}$ are duals of the bases generated by the analysis functions ϕ and w. Given $\{V_i\}$ and $\{W_i\}$ as defined in Section 4, this duality can be seen in the relations

$$V_i \perp \tilde{W}_i$$

and

$$\tilde{V}_i \perp W_i.$$

Since the dual wavelets at level i are orthogonal to the primal scaling functions at the same level, i.e.,

$$\langle \tilde{w}(2^i t - k), \phi(2^i t - k) \rangle = 0, \text{[1]}$$

$\tilde{\phi}$ and $\tilde{w}$ give us a way to compute the coefficients of ϕ and w much in the same way that property (5) allows us to compute the a_i's in (4). To see this, consider the following analysis of an input signal f:

$$f = \sum_k a_k \phi(t-k) + \sum_{i,k} b_{ik} w(2^i t - k)$$

We can obtain the a_k's and b_{ik}'s as follows:

$$\begin{aligned} a_k &= \langle f, \tilde{\phi}(t-k) \rangle \\ b_{ik} &= \langle f, \tilde{w}(2^i t - k) \rangle \end{aligned}$$

6 Where to Enforce Which Conditions

The desirable conditions we started out with were formulated for single wavelet bases, whereas we are now dealing with two wavelet bases — a primary and a dual. How does that change the way we enforce the conditions?

Symmetry is not a problem: an immediate consequence of the alternating flip is that each filter in the primary bank is symmetric with respect to the corresponding filter in the dual bank.

[1] In fact $\langle \tilde{w}(2^i t - k), \phi(2^j t - k) \rangle = 0$ for any $j \leq i$.

Good approximation, in the case of a single wavelet, required a large number of vanishing moments. Now that we have two distinct wavelets w and $\tilde{w}$, which one should have many vanishing moments? The same argument applies as in the single-basis case of Lecture #3; there it was shown that a coefficient a_i would be small if the basis function f_i had many vanishing moments. The derivation hinged on the condition

$$a_i = \langle f, f_i \rangle$$

in order to arrive at the expression

$$a_i = \int (t - t_0)^P R(t) f_i(t) dt$$

and, ultimately, at a bound on the magnitude of a_i. With our dual wavelets, a similar derivation would start out with

$$b_{ik} = \langle f, \tilde{w}(2^i t - k) \rangle.$$

Immediately we see that the dual wavelet $\tilde{w}$ (as opposed to the primal w) is the one that should have many vanishing moments.

As for smoothness, recall that when a function gets projected onto a basis and some of the projected elements are discarded, the truncated projection takes on the local properties of the basis — in particular, smoothness or jaggedness. For instance, if a function

$$f = \sum_{i=0}^{\infty} a_i f_i$$

is approximated by

$$\tilde{f} = \sum_{i=0}^{J} a_i f_i,$$

$\tilde{f}$ takes on the smoothness properties of the f_i's. In compression applications, truncation is applied after the analysis stage, which means that the truncated terms come from the primary basis. Therefore we should strive for a smooth analysis filter. In other words, the smoothness condition should be enforced on the primal w.

7 Spline Wavelets

We conclude with a simple example of a biorthogonal wavelet basis. As we have seen in the beginning, orthogonality prevents us from constructing finitely supported scaling functions; in particular, we cannot use spline basis functions as scaling functions. Recall our discussion of the cascade algorithm: a linear spline satisfies the scaling equation with coefficients 1/4, 1/2 and 1/4. In addition to symmetry, splines have one more very nice feature: out of all scaling functions with given support, they have maximal smoothness.

This is exactly one of our requirements for the scaling functions/wavelets of the synthesis filter bank.

In the simplest spline filter bank, F_0 is the linear spline filter 1/4, 1/2, 1/4. Recall that we can get the analysis filter H_1 by alternating flip. This leaves us one filter to choose, say, H_0. The simplest filter satisfying Equation (2) has just one coefficient $h(0) = 1$. This completely determines the filter. However, it is clear that the analysis filters are quite bad: the low-pass part of the signal is just subsampled version of the filter. Note that the high-pass filter of the synthesis filter has one vanishing moment. By increasing the support of the analysis filters, we can improve them, ensuring existence of corresponding wavelet basis and greater number of vanishing moments. For example, if the support size is increased to 4, we obtain coefficients $-1/8$, $1/4$, $3/4$, $1/4$, $-1/8$ for the analysis filter H_0. This is so-called 5/3 spline filter bank (the numbers refer to the number coefficients in the analysis and synthesis low-pass filters.)

This filter bank is shown below.