# Sampling and Reconstruction

The sampling and reconstruction process

- Real world: continuous
- Digital world: discrete

Basic signal processing

- Fourier transforms
- The convolution theorem
- The sampling theorem

Aliasing and antialiasing

- Uniform supersampling
- Nonuniform supersampling

# Imagers = Signal Sampling

All imagers convert a continuous image to a discrete sampled image by integrating over the active "area" of a sensor.

$$R=\int_T\iint_\Omega\iint_A L(x,\omega,t)P(x)S(t)\cos\theta\, dA\, d\omega\, dt$$

Examples:

- Retina: photoreceptors
- CCD array

Virtual CG cameras do not integrate,

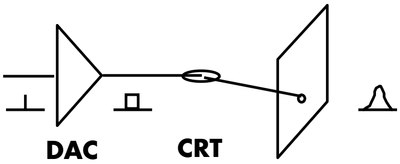
they simply sample radiance along rays …

## Displays = Signal Reconstruction

All physical displays recreate a continuous image from a discrete sampled image by using a finite sized source of light for each pixel.

Examples:

- DACs: sample and hold
- Cathode ray tube: phosphor spot and grid



## Sampling in Computer Graphics

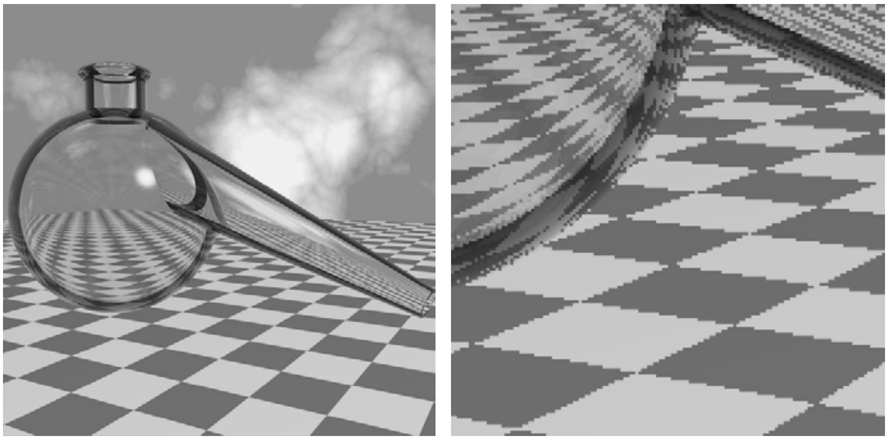
Artifacts due to sampling - Aliasing

- Jaggies
- Moire
- Flickering small objects
- Sparkling highlights
- Temporal strobing

Preventing these artifacts - Antialiasing

# Jaggies

**Retort sequence by Don Mitchell**



**Staircase pattern or jaggies**

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# Basic Signal Processing

# Fourier Transforms

**Spectral representation treats the function as a weighted sum of sines and cosines**

**Each function has two representations**

- **Spatial domain - normal representation**
- **Frequency domain - spectral representation**

**The *Fourier transform* converts between the spatial and frequency domain**

**Spatial Domain**

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \Rightarrow$$

$$\Leftarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega \Leftarrow$$

**Frequency Domain**

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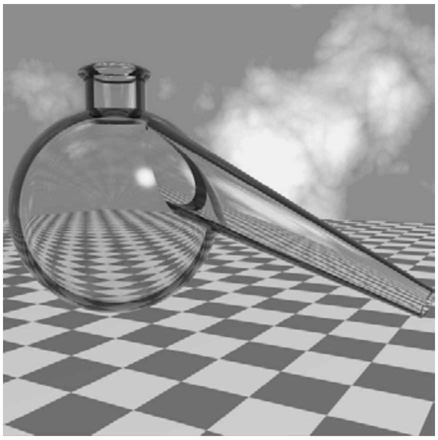
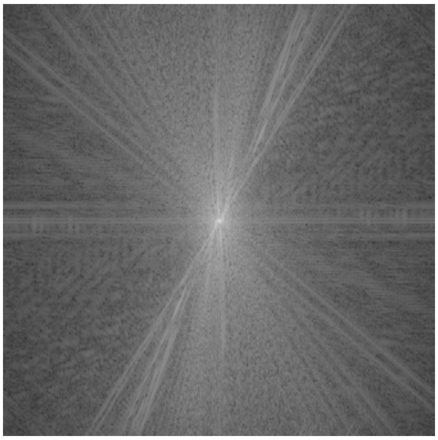
# Spatial and Frequency Domain

**Spatial Domain**



**Frequency Domain**



## Convolution

***Definition***

$$h(x) = f \otimes g = \int f(x')g(x-x')\,dx'$$

***Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.**

$$f \otimes g \leftrightarrow F \times G$$

***Symmetric Theorem*: Multiplication in the space domain is equivalent to convolution in the frequency domain.**

$$f \times g \leftrightarrow F \otimes G$$

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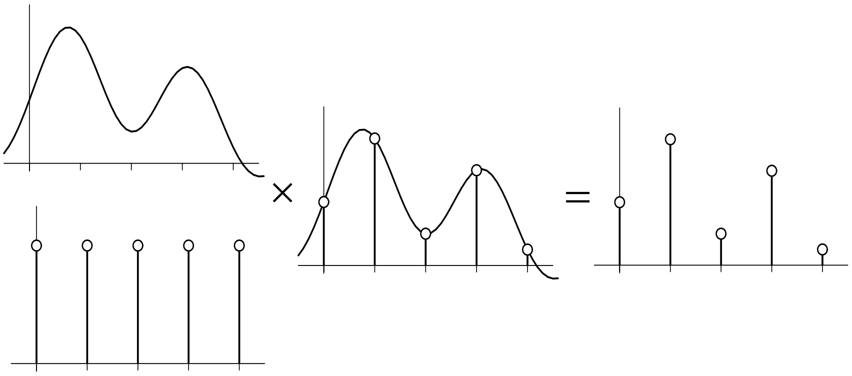
# The Sampling Theorem

## Sampling: Spatial Domain



$$\mathrm{III}(x) = \sum_{n=-\infty}^{n=\infty} \delta(x - nT)$$

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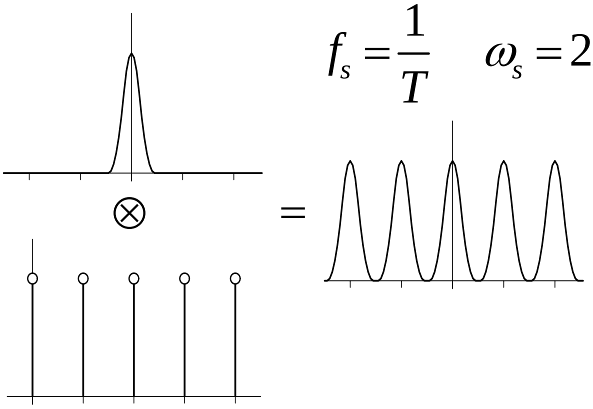
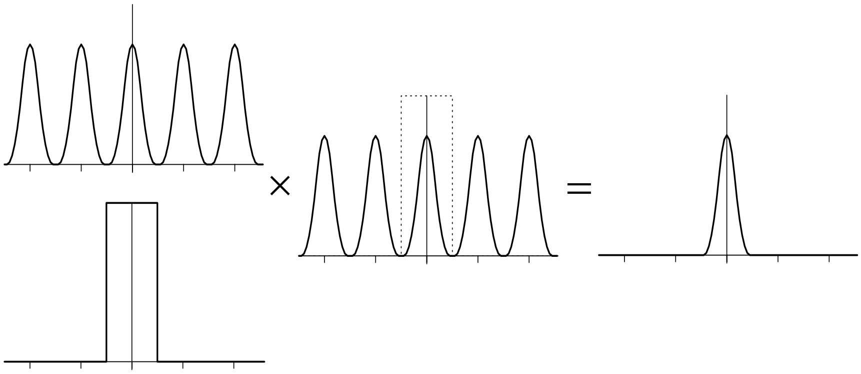
## Sampling: Frequency Domain

$$f_s = \frac{1}{T} \qquad \omega_s = 2\pi f_s = \frac{2\pi}{T}$$



$$\mathrm{III}(\omega) = \sum_{n=-\infty}^{n=\infty} \delta(\omega - n\omega_s)$$

## Reconstruction: Frequency Domain



$$\text{II}(x) = \begin{cases} 1 & |x| \leq \tfrac{1}{2} \\ 0 & |x| > \tfrac{1}{2} \end{cases}$$

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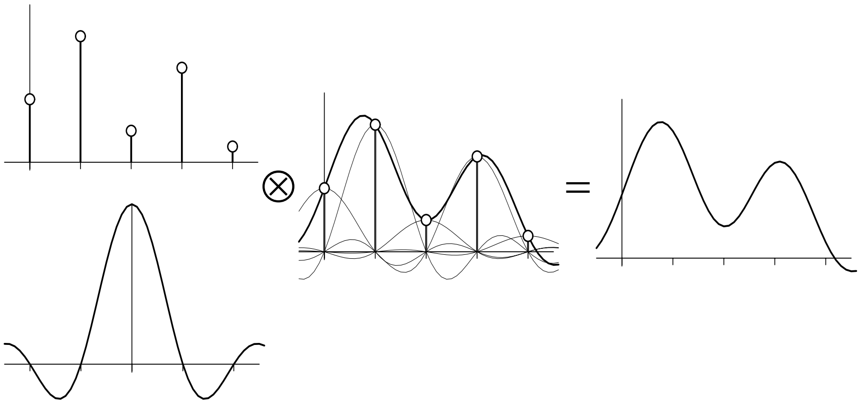
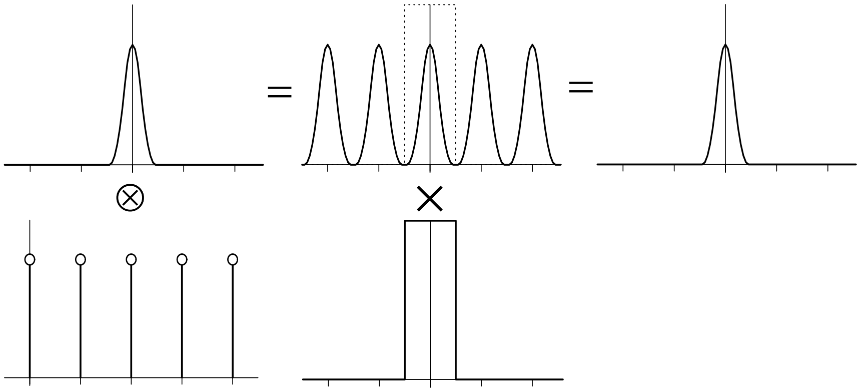
## Reconstruction: Spatial Domain



$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

# Sampling and Reconstruction



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# Sampling Theorem

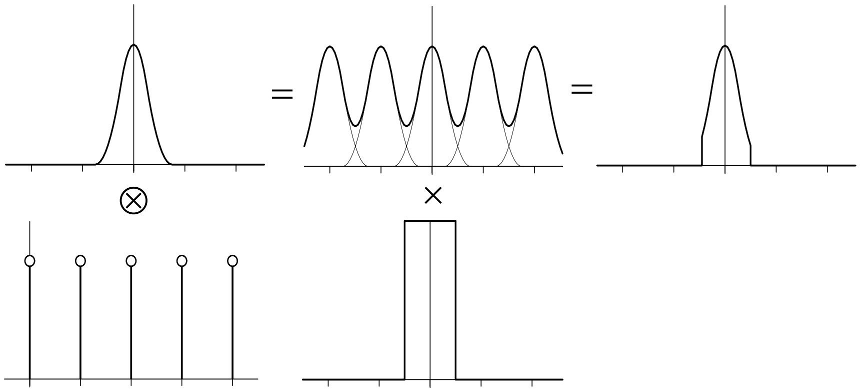
This result if known as the Sampling Theorem and is due to Claude Shannon who first discovered it in 1949

> A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the Sampling frequency

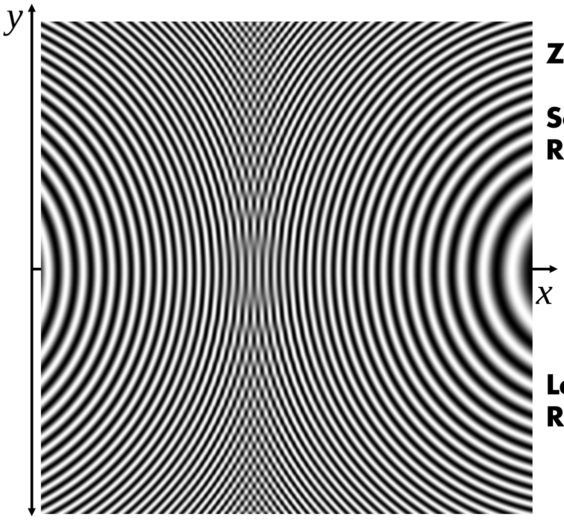
For a given bandlimited function, the rate at which it must be sampled is called the ***Nyquist Frequency***

# Aliasing

## Undersampling: Aliasing



# Sampling a "Zone Plate"



**Zone plate:** $\sin x^2 + y^2$

**Sampled at 128x128**
**Reconstructed to 512x512**
**Using a 30-wide**
**Kaiser windowed sinc**

**Left rings: part of signal**
**Right rings: prealiasing**

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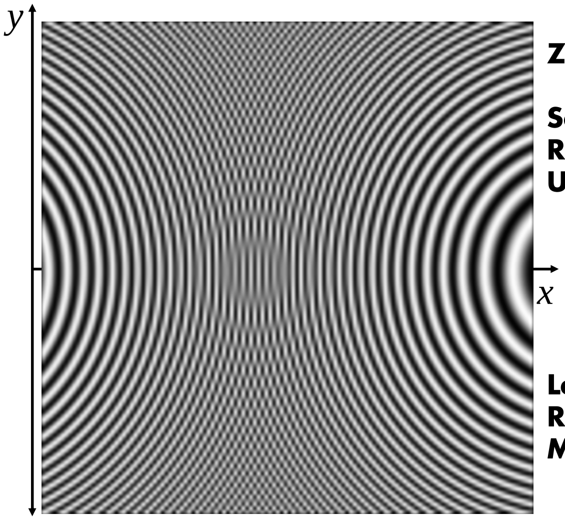
# Ideal Reconstruction

Ideally, use a perfect low-pass filter - the sinc function - to bandlimit the sampled signal and thus remove all copies of the spectra introduced by sampling

Unfortunately,

- The sinc has infinite extent and we must use simpler filters with finite extents. Physical processes in particular do not reconstruct with sincs
- The sinc may introduce ringing which are perceptually objectionable

## Sampling a "Zone Plate"



**Zone plate:** $\sin x^2 + y^2$

**Sampled at 128x128**
**Reconstructed to 512x512**
**Using optimal cubic**

**Left rings: part of signal**
**Right rings: prealiasing**
**Middle rings: postaliasing**

## Mitchell Cubic Filter

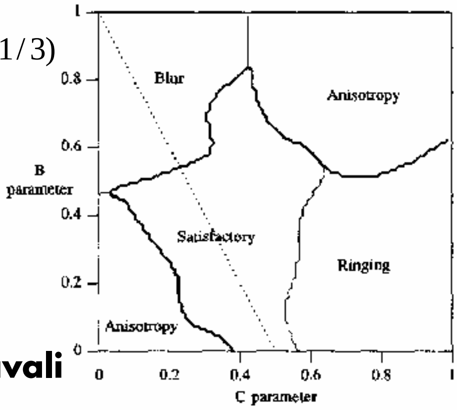
$$h(x) = \frac{1}{6}\begin{cases} (12-9B-6C)x^3 + (-18+12B+6C)x^2 + (6-2B) & |x|<1 \\ (-B-6C)x^3 + (6B+30C)x^2 + (-12B-48C)x + (8B+24C) & 1<|x|<2 \\ 0 & \textit{otherwise} \end{cases}$$

Good: $(1/3, 1/3)$

**Properties:**

$$\sum_{n=-\infty}^{n=\infty} h(x) = 1$$

B-spline: $(1,0)$
Catmull-Rom: $(0,1/2)$



**From Mitchell and Netravali**

# Antialiasing

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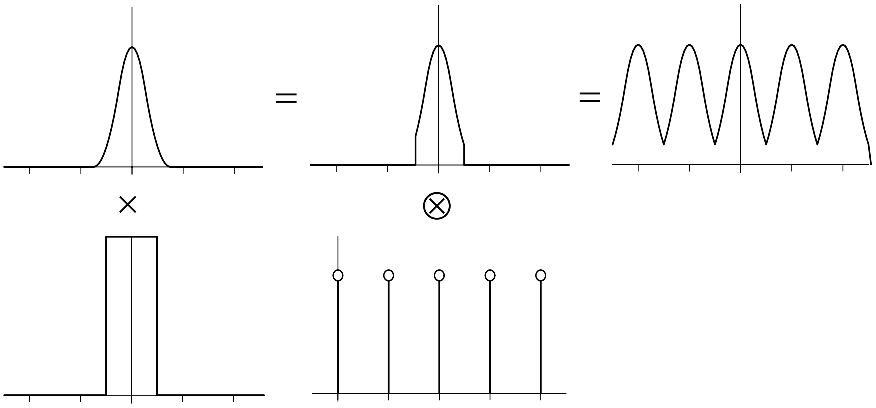
# Antialiasing

**Antialiasing = Preventing aliasing**

1. **Analytically prefilter the signal**
    - **Solvable for points, lines and polygons**
    - **Not solvable in general**
      **e.g. procedurally defined images**
2. **Uniform supersampling and resample**
3. **Nonuniform or stochastic sampling**

# Antialiasing by Prefiltering



**Frequency Space**

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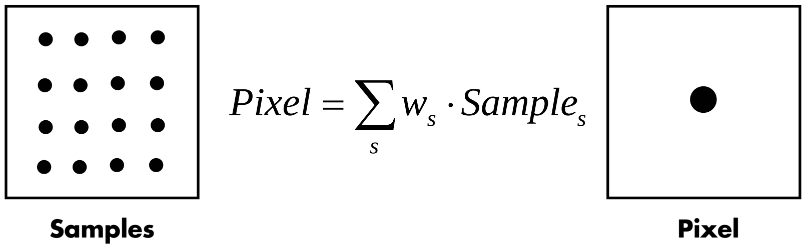
# Uniform Supersampling

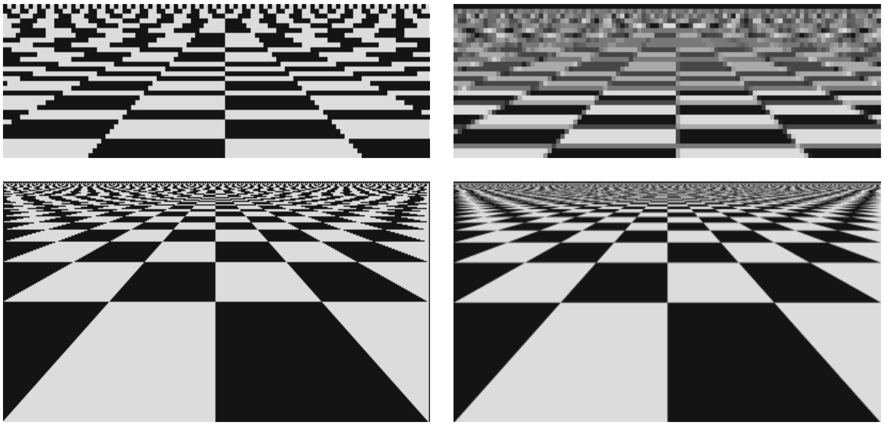
Increasing the sampling rate moves each copy of the spectra further apart, potentially reducing the overlap and thus aliasing

Resulting samples must be resampled (filtered) to image sampling rate



$$Pixel = \sum_s w_s \cdot Sample_s$$

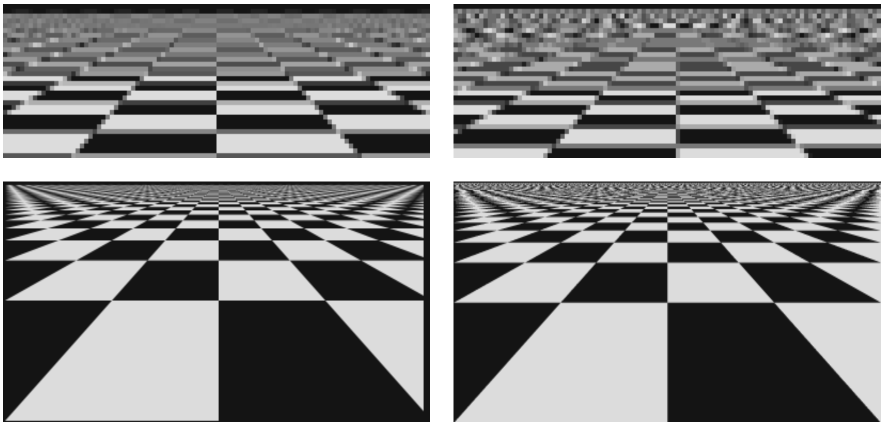
## Point vs. Supersampled



**Point** **4x4 Supersampled**

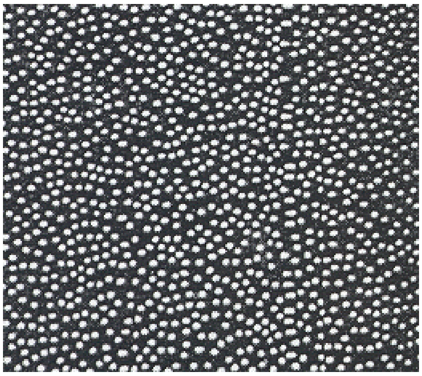
Checkerboard sequence by Tom Duff

## Analytic vs. Supersampled

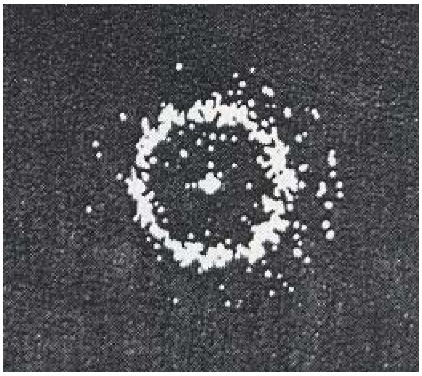


**Exact Area** **4x4 Supersampled**

# Distribution of Extrafoveal Cones



**Monkey eye cone distribution**



**Fourier transform**

**Yellot theory**

- **Aliases replaced by noise**
- **Visual system less sensitive to high freq noise**

# Non-uniform Sampling

**Intuition**

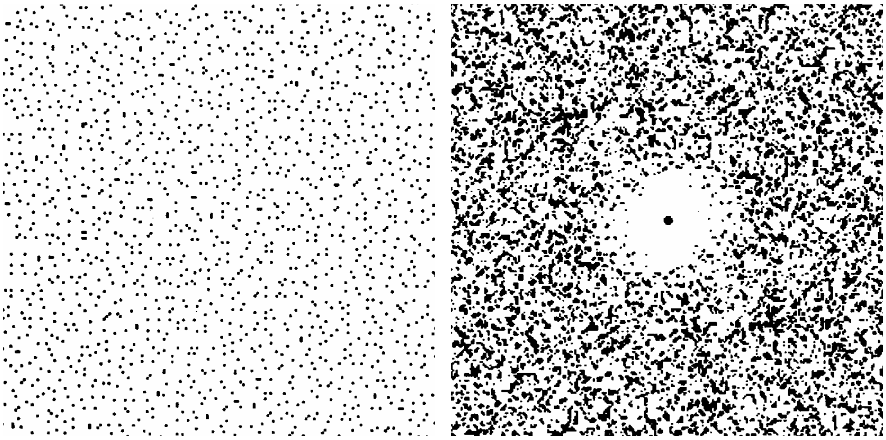
**Uniform sampling**

- **The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes**
- **Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)**
- **Aliases are coherent, and very noticable**

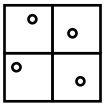
**Non-uniform sampling**

- **Samples at non-uniform locations have a different spectrum; a single spike plus noise**
- **Sampling a signal in this way converts aliases into broadband noise**
- **Noise is incoherent, and much less objectionable**

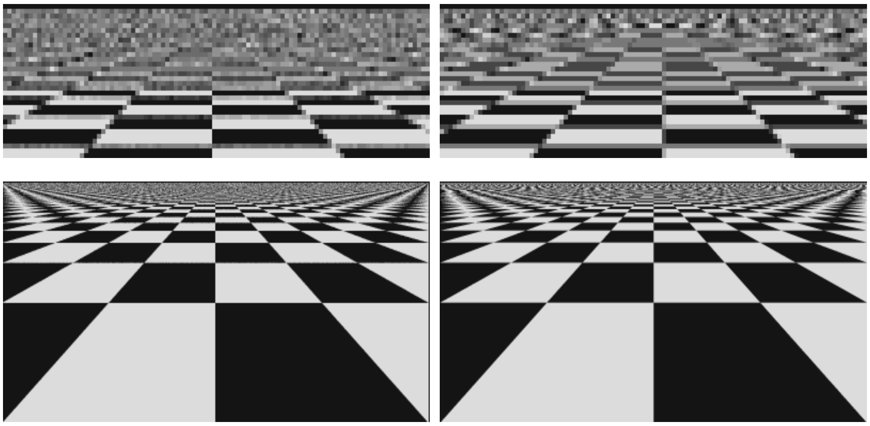
# Jittered Sampling



**Add uniform random jitter to each sample**



# Jittered vs. Uniform Supersampling



**4x4 Jittered Sampling** **4x4 Uniform**

# Analysis of Jitter

**Non-uniform sampling**

$$s(x) = \sum_{n=-\infty}^{n=\infty} \delta(x - x_n)$$

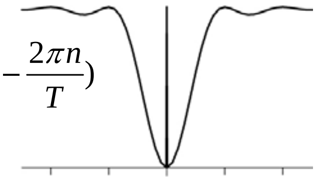
$$x_n = nT + j_n$$

**Jittered sampling**

$$j_n \sim j(x)$$

$$j(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$J(\omega) = \operatorname{sinc} \omega$$



$$S(\omega) = \frac{1}{T}\left[1 - |J(\omega)|^2\right] + \frac{2\pi}{T^2}|J(\omega)|^2 \sum_{n=-\infty}^{n=-\infty} \delta(\omega - \frac{2\pi n}{T})$$

$$= \frac{1}{T}\left[1 - \operatorname{sinc}^2 \omega\right] + \delta(\omega)$$

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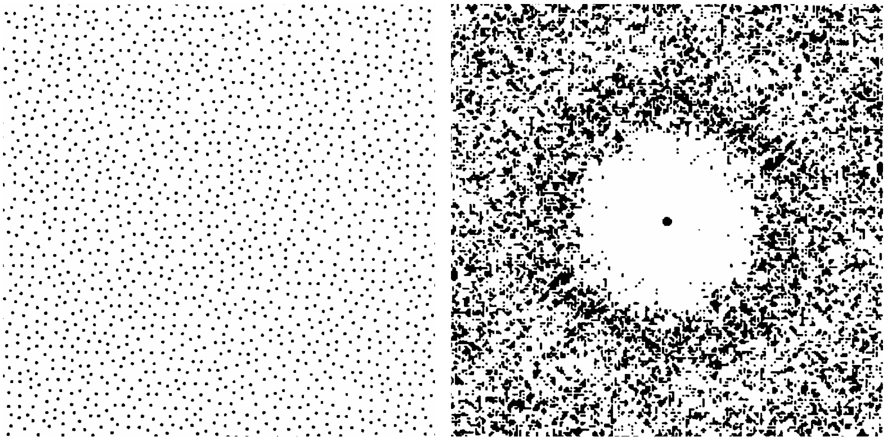
# Poisson Disk Sampling



**Dart throwing algorithm**