Monte Carlo I

Previous lecture

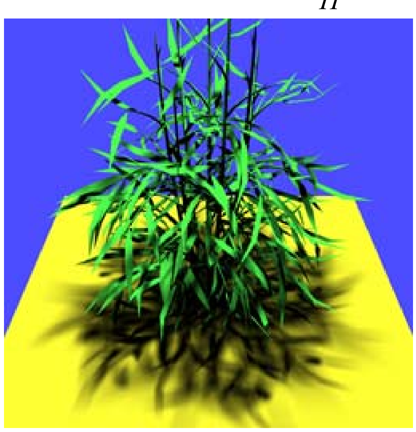
- Analytical illumination formula

This lecture

- Review random variables and probability
- Monte Carlo integration
- Sampling from distributions
- Sampling from shapes
- Variance and efficiency

Lighting and Soft Shadows

$$E(x) = \int_{H^2} L_i(x,\omega)\cos\theta\, d\omega$$



Challenges

- Visibility and blockers
- Varying light distribution
- Complex source geometry

Source: Agrawala. Ramamoorthi, Heirich, Moll, 2000

Monte Carlo Algorithms

Advantages

- Easy to implement
- Easy to think about (but be careful of statistical bias)
- Robust when used with complex integrands and domains (shapes, lights, ...)
- Efficient for high dimensional integrals
- Efficient solution method for a few selected points

Disadvantages

- Noisy
- Slow (many samples needed for convergence)

Random Variables

X is chosen by some random process

$X \sim p(x)$ probability distribution function (PDF)

CS348B Lecture 6 Pat Hanrahan, Spring 2004

Discrete Probability Distributions

Discrete events X_i **with probability** p_i

$$p_i \geq 0 \qquad \sum_{i=1}^{n} p_i = 1$$

Cumulative PDF

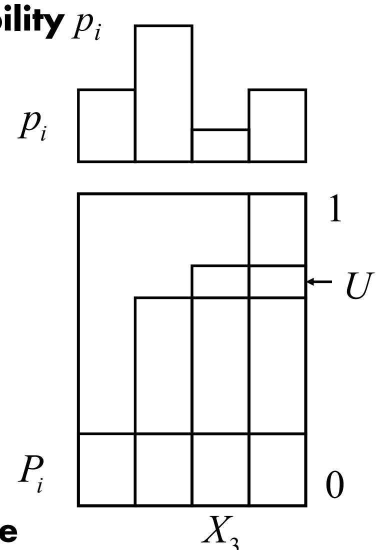
$$P_j = \sum_{i=1}^{j} p_i$$

Construction of samples

To randomly select an event,

Select X_i **if** $P_{i-1} < U \leq P_i$

Uniform random variable



Continuous Probability Distributions

PDF $p(x)$

$$p(x) \geq 0$$

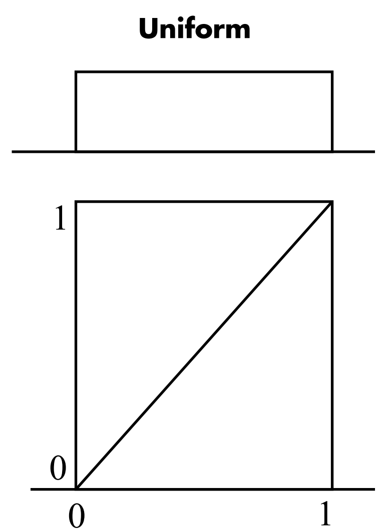
CDF $P(x)$

$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$



Sampling Continuous Distributions

Cumulative probability distribution function

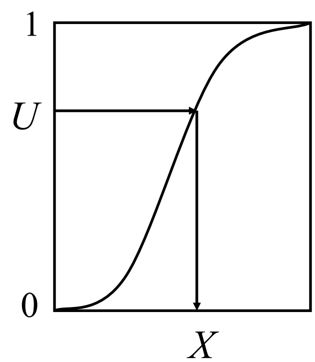
$$P(x) = \Pr(X < x)$$

Construction of samples

Solve for $X=P^{-1}(U)$

Must know:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$



Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick

$$Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$$

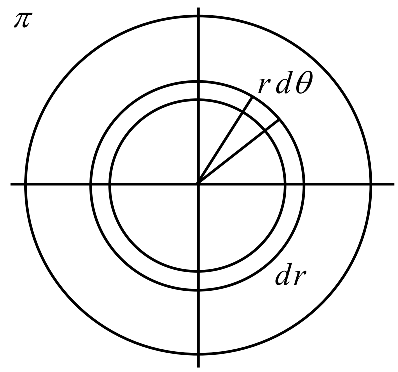
$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

Sampling a Circle

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{\pi} r\,dr\,d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

$$p(r,\theta) = p(r)p(\theta)$$



$$p(\theta) = \frac{1}{2\pi}$$

$$P(\theta) = \frac{1}{2\pi}\theta$$

$$\theta = 2\pi U_1$$

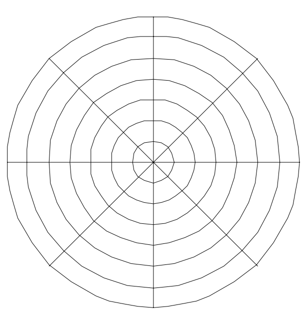
$$p(r) = 2r$$

$$P(r) = r^2$$

$$r = \sqrt{U_2}$$

Sampling a Circle

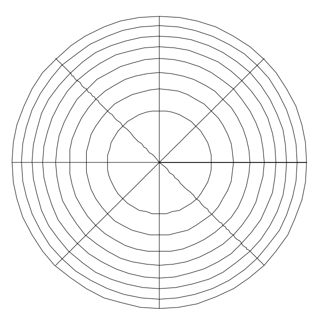
WRONG ≠ Equi-Areal



$$\theta = 2\pi U_1$$

$$r = U_2$$

RIGHT = Equi-Areal



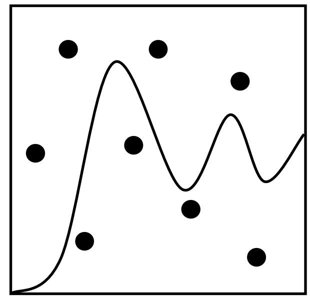
$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

Rejection Methods

$$I = \int_0^1 f(x)\,dx$$

$$= \iint_{y<f(x)} dx\,dy$$



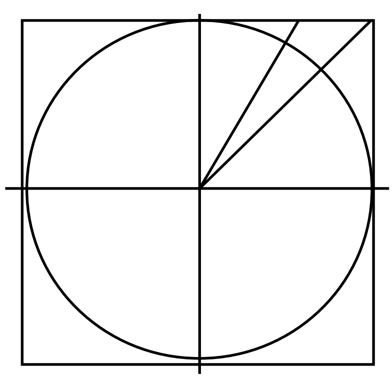
Algorithm

Pick U_1 **and** U_2

Accept U_1 **if** $U_2 < f(U_1)$

Wasteful? Efficiency = Area / Area of rectangle

Sampling a Circle: Rejection



```
do {
   X=1-2*U1
   Y=1-2*U2
while( X²+ Y² >1 )
```

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

Monte Carlo Integration

Definite integral $I(f) \equiv \int_0^1 f(x)\, dx$

Expectation of f $E[f] \equiv \int_0^1 f(x)\, p(x)\, dx$

Random variables $X_i \sim p(x)$

$Y_i = f(X_i)$

Estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} Y_i$

Unbiased Estimator

$$\boxed{E[F_N] = I(f)}$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$
$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x) p(x)\, dx$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x) dx$$
$$= \int_0^1 f(x)\, dx$$

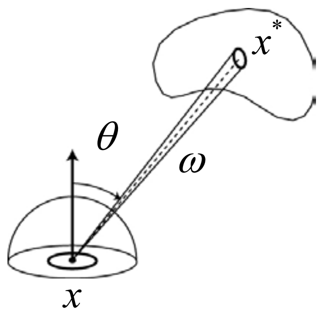
Assume uniform probability distribution for now

Properties

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$
$$E[aY] = aE[Y]$$

Direct Lighting – Directional Sampling

$$E(x) = \int_{\Omega} L(x,\omega)\cos\theta\, d\omega$$



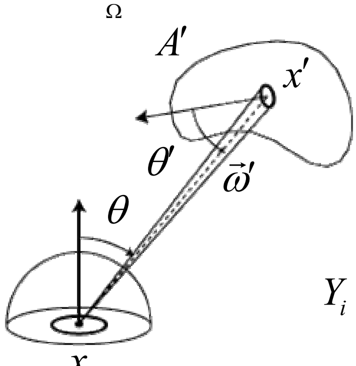
Ray intersection $x^*(x,\omega)$

Sample ω **uniformly by** Ω

$$Y_i = L(x^*(x,\omega_i),-\omega_i)\cos\theta\, 2\pi$$

Direct Lighting – Area Sampling

$$E(x) = \int_{\Omega} L_i(x,\omega)\cos\theta\, d\omega = \int_{A'} L_o(x',\omega')V(x,x')\frac{\cos\theta\cos\theta'}{|x-x'|^2}\,dA'$$



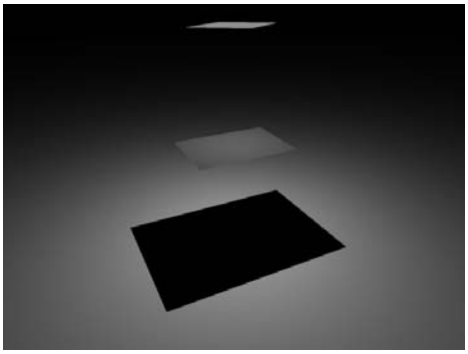
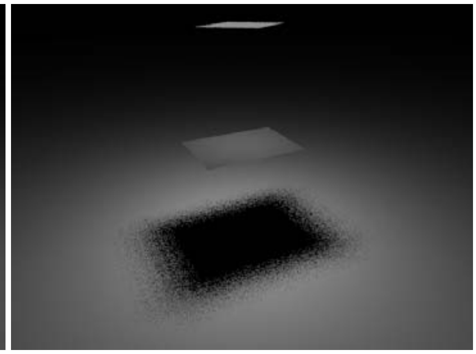
Ray direction $\omega' = x - x'$

Sample x' **uniformly by** A'

$$Y_i = L_o(x_i',\omega_i')\,V(x,x_i')\frac{\cos\theta\,\cos\theta_i'}{|x-x_i'|^2}\,A$$

$$V(x,x') = \begin{cases} 0 & \neg visible \\ 1 & visible \end{cases}$$

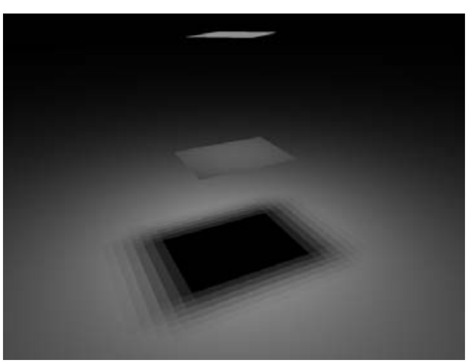
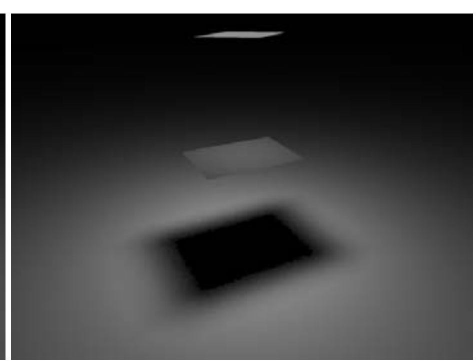
Examples

4 eye rays per pixel
1 shadow ray per eye ray

Fixed | **Random**

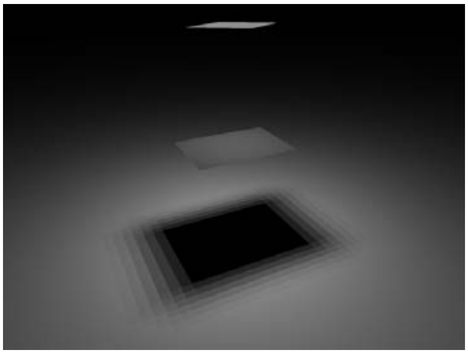
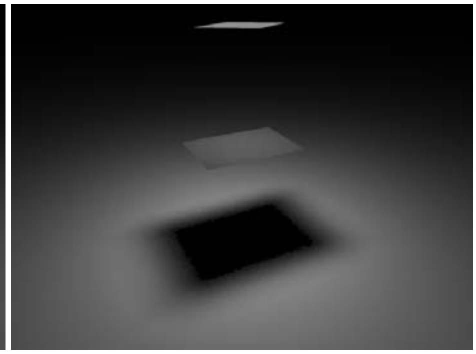
Examples

4 eye rays per pixel
16 shadow rays per eye ray

Uniform grid | **Stratified random**

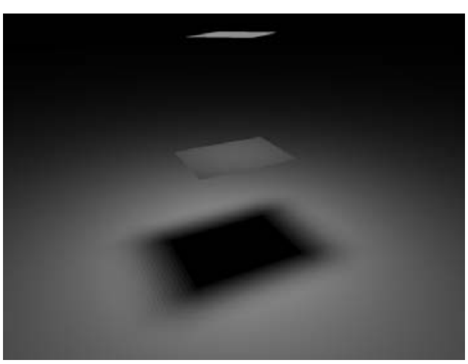
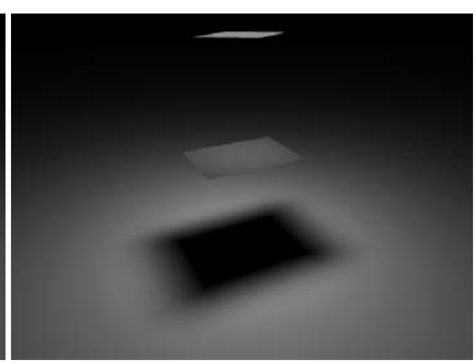
Examples

4 eye rays per pixel
64 shadow rays per eye ray

Uniform grid **Stratified random**

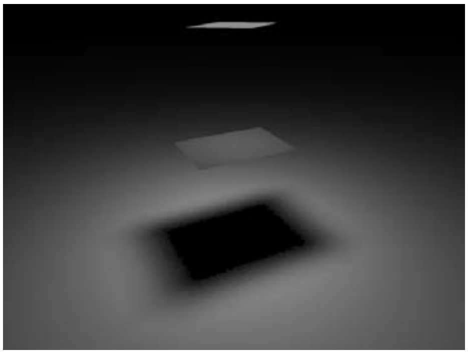
Examples

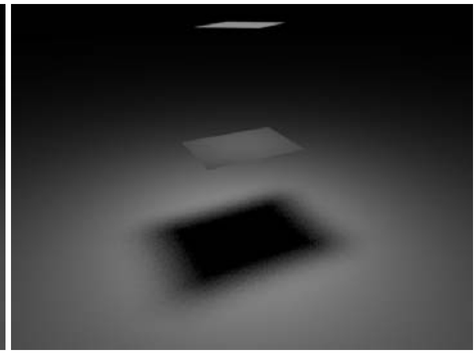
4 eye rays per pixel
100 shadow rays per eye ray

Uniform grid **Stratified random**

Examples



4 eye rays per pixel
16 shadow rays per eye ray



64 eye rays per pixel
1 shadow ray per eye ray

Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2 - 2YE[Y] + E[Y]^2]$$
$$= E[Y^2] - E[Y]^2$$

Properties

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$

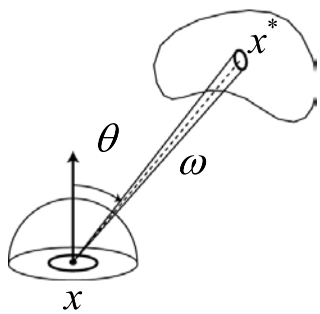
$$V[aY] = a^2 V[Y]$$

Variance decreases with sample size

$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N} V[Y]$$

Direct Lighting – Directional Sampling

$$E(x) = \int_{\Omega} L(x,\omega)\cos\theta \, d\omega$$



Ray intersection $x^*(x,\omega)$

Sample ω **uniformly by** Ω

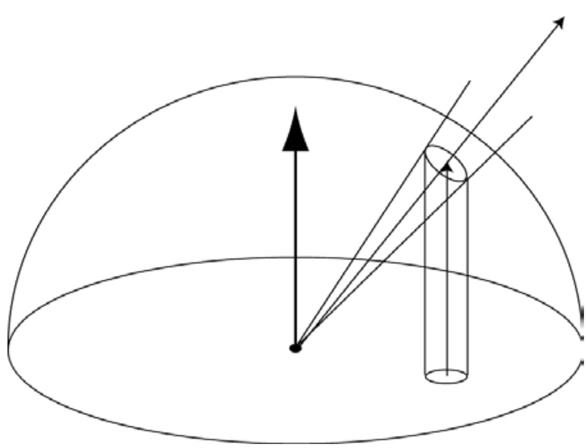
$$Y_i = L(x^*(x,\omega_i), -\omega_i)\cos\theta \, 2\pi$$

Sample ω **uniformly by** $\tilde{\Omega}$

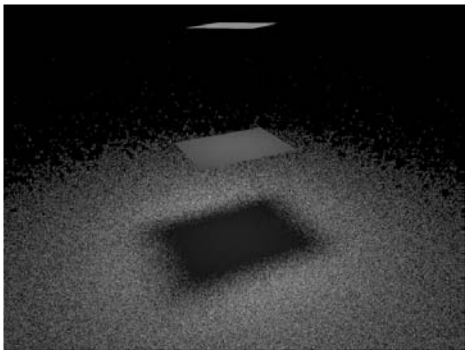
$$Y_i = L(x^*(x,\omega_i), -\omega_i)\,\pi$$

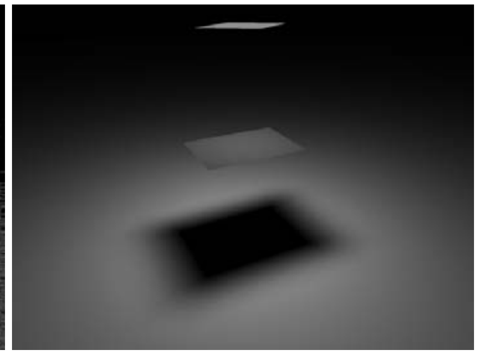
Sampling Projected Solid Angle

Generate cosine weighted distribution



Examples





Projected solid angle

4 eye rays per pixel
100 shadow rays

Area

4 eye rays per pixel
100 shadow rays

Variance Reduction

Efficiency measure

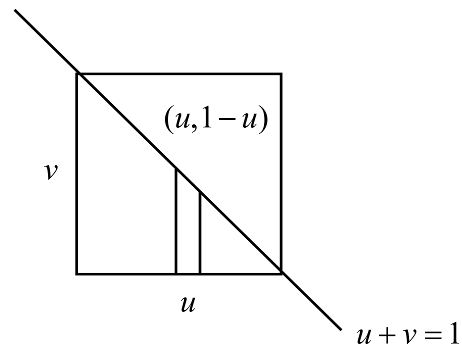
$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

Techniques

- **Importance sampling**
- **Sampling patterns: stratified, ...**

Sampling a Triangle

$$u \ge 0$$
$$v \ge 0$$
$$u + v \le 1$$



$$A = \int_0^1 \int_0^{1-u} dv\, du = \int_0^1 (1-u)\, du = -\left.\frac{(1-u)^2}{2}\right|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

Sampling a Triangle

Here u and v are not independent! $p(u,v) = 2$

Conditional probability

$$p(u) \equiv \int p(u,v)\, dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(u)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\, du \quad = (1-u_0)^2$$

$$u_0 = 1 - \sqrt{U_1}$$

$$p(v \mid u) = \frac{1}{(1-u)}$$

$$P(v_0 \mid u_0) = \int_0^{v_o} p(v \mid u_0)\, dv \quad = \int_0^{v_o} \frac{1}{(1-u_0)} dv \quad = \frac{v_0}{(1-u_0)}$$

$$v_0 = \sqrt{U_1} U_2$$