

Rendering Operators

Formal solution to the rendering equation

$$L(x, \mathbf{w}) = \sum_{k=0}^{\infty} \int_{M^2} \int_{M^2} \dots \int_{M^2} K(x_0, x_1, x_2) \cdots K(x_k, x, \mathbf{w}) L_e(x_0, x_1) dA_0 dA_1 \dots dA_k$$

The measurement equation

$$R = \int_t \int_w \int_A \int L(x, \mathbf{w}, t) P(x) S(t) \cos \mathbf{q} \cdot dA d\mathbf{w} dt$$

Lots of darn integrals:

- Integrate over time: Motion blur
- Integrate over pixel x, y : Antialiasing
- Integrate over lens u, v : Depth of field
- Integrate over paths with k bounces: Light transport

Overview

Today

- Direct Lighting
- Sampling distributions
- Sampling shapes
- Importance sampling
- Multiple importance sampling

Thursday

- Path tracing
- Bidirectional ray tracing
- Density estimation and photon tracing

Monte Carlo Integration

Definite integral

$$I = \int f(x) dx$$

Expectation of f

$$E[f] = \int f(x) p(x) dx$$

Estimators

$$X_i \sim p(x)$$

$$Y_i = f(X_i)$$

$$E[f] = \frac{1}{N} \sum_{i=1}^N Y_i$$

Monte Carlo Algorithms

Sampling

- Signal processing view (aliases)
- Statistical view (variance)
- Quasi-random sampling (discrepancy)

Advantages

- Easy to think about (but deceptive!)
- Easy to implement
- Robust; point sampling process
- Efficient for high dimensional space, complex fns
- Efficient solution method for a few selected points

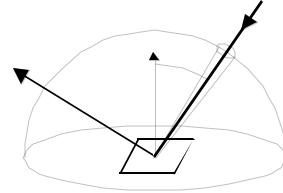
Direct Illumination

General case

$$L_r(x, \mathbf{w}_r) = \int_{H^2} f_r(x, \mathbf{w}_i \rightarrow \mathbf{w}_r) L_i(x, \mathbf{w}_i) \cos \mathbf{q}_i d\mathbf{w}_i$$

Point light

$$L_r(x, \mathbf{w}_r) = \sum_{j=1}^n f_r(x, \mathbf{w}_j \rightarrow \mathbf{w}_r) \frac{\Phi_j}{4\pi \|x - x_j\|^2} \cos \mathbf{q}_j$$



Area light

$$L_r(x, \mathbf{w}_r) = \int_{M^2} f_r(x, (x - x') \rightarrow \mathbf{w}_r) L_o(x', (x - x')) \frac{\cos \mathbf{q} \cos \mathbf{q}'}{\|x - x'\|^2} dA'$$

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Discrete Probability Distributions

Discrete events \mathbf{X}_i with probability p_i

$$p_i \geq 0$$

$$\sum_{i=1}^n p_i = 1$$

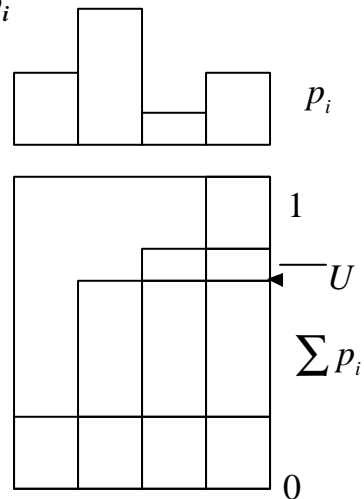
Construction of samples

To randomly select an event,

Select \mathbf{X}_i if

$$p_1 + \dots + p_{i-1} < U \leq p_1 + \dots + p_{i-1} + p_i$$

Uniform random variable



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Examples

Sampling a set of point light sources

Associate a probability with each light

$$p_i = \frac{\Phi_i}{\sum_{j=1}^n \Phi_j}$$

Choosing between reflection and transmission

$$p_r = \frac{F_r}{F_r + F_t} \quad p_t = \frac{F_t}{F_r + F_t}$$

Continuous Probability Distributions

Cumulative probability distribution function

$$P(x) = \Pr(X_i < x)_b$$

$$\Pr(\mathbf{a} \leq X_i \leq \mathbf{b}) = \int_a^b p(x) dx = P(\mathbf{b}) - P(\mathbf{a})$$

Probability density function

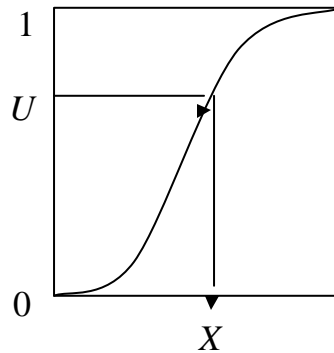
$$p(x) = \frac{dP(x)}{dx}$$

Construction of samples

$$\text{Solve for } \mathbf{X} = \mathbf{P}^{-1}(U)$$

Must know:

1. The integral of $p(x)$
2. The inverse function $\mathbf{P}^{-1}(x)$



Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) = \sqrt[n+1]{U}$$

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

Trick: $Y = \max(U_1, U_2, \dots, U_n, U_{n+1})$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U_i < x) = x^{n+1}$$

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Sampling a Circle

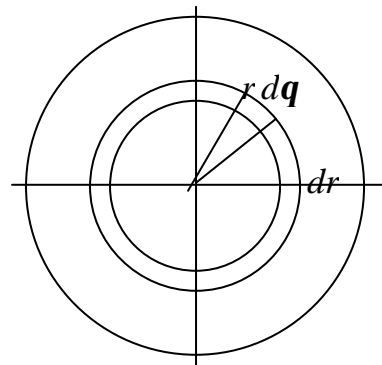
$$A = \int_0^{2p} \int_0^1 r dr dq = \left(\frac{r^2}{2} \right) (2p) = p$$

$$p(r, q) dr dq = \frac{1}{p} r dr dq$$

$$P(r, q) = \frac{1}{2p} r^2 q$$

$$q = 2p U_1$$

$$r = \sqrt{U_2}$$

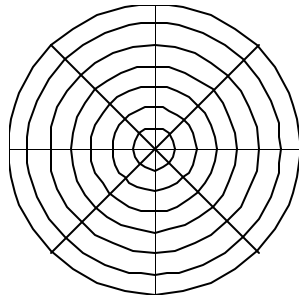


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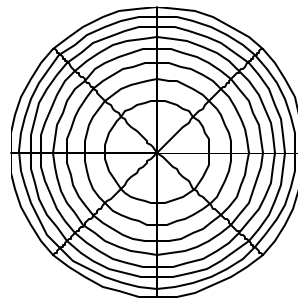
Sampling a Circle

Strata



$$r = U_1$$
$$q = 2pU_2$$

Equi-Areal



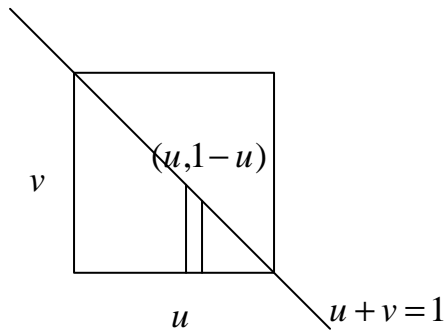
$$r = \sqrt{U_1}$$
$$q = 2pU_2$$

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Sampling a Triangle

$$u \geq 0$$
$$v \geq 0$$
$$u + v \leq 1$$



$$A = \int_0^1 \int_0^{1-u} dv du = \int_0^1 (1-u) du = -\frac{(1-u)^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$p(u, v) = 2$$

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Sampling a Triangle

Here u and v are not independent

$$p(u, v) = 2$$
$$p(u) = 2 \int_0^{1-u} dv = 2(1-u)$$
$$p(v|u) = \frac{1}{(1-u)}$$

$$P(u_1) = \int_0^{u_0} 2(1-u) du = (1-u_0)^2 \quad u_0 = 1 - \sqrt{U_1}$$

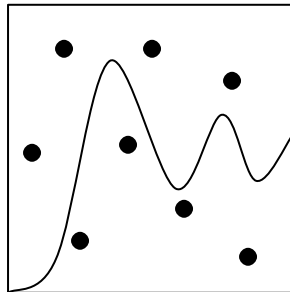
$$P(v|u) = \int_0^{v_0} \frac{1}{(1-u_0)} dv = \frac{v_0}{(1-u_0)} \quad v_0 = \sqrt{U_1} U_2$$

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Rejection Methods

$$I = \int_0^1 f(x) dx$$
$$= \iint_{y < f(x)} dx dy$$



Algorithm

Pick U_1 and U_2

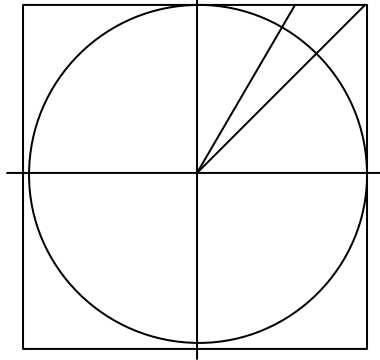
Accept U_1 if $U_2 < f(U_1)$

Wasteful? Efficiency = Area / Area of rectangle

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Sampling a Circle: Rejection



do {
 $X=1-2U_1$
 $Y=1-2U_2$
while($X^2+Y^2 > 1$)

May be used to pick random 2D directions
Circle techniques may also be applied to the sphere

Biassing

Biassing the sampling process

$$X_i \sim p(x) \quad Y_i = \frac{f(X_i)}{p(X_i)}$$

$$\begin{aligned} E[Y_i] &= E\left[\frac{f(X_i)}{p(X_i)}\right] \\ &= \int \left[\frac{f(X_i)}{p(X_i)}\right] p(x) dx \\ &= \int f(x) dx \\ &= I \end{aligned}$$

Importance Sampling

Variance

$$V[f] = E[f^2] - E^2[f]$$

$$E[Y_i^2] = \int \left[\frac{f(X_i)}{p(X_i)} \right]^2 p(x) dx$$

Zero variance biasing

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$V[\tilde{f}^2] = 0$$

$$E[\tilde{f}^2] = \int \left[\frac{f(X_i)}{\tilde{p}(X_i)} \right]^2 \tilde{p}(x) dx$$

$$= \int E^2[f] \tilde{p}(x) dx$$

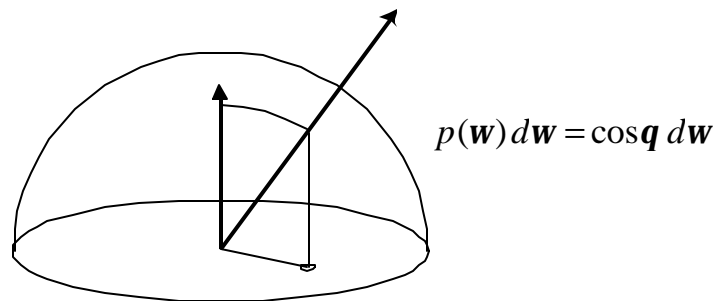
$$= E^2[f]$$

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Diffuse BRDF

Generate cosine weighted distribution



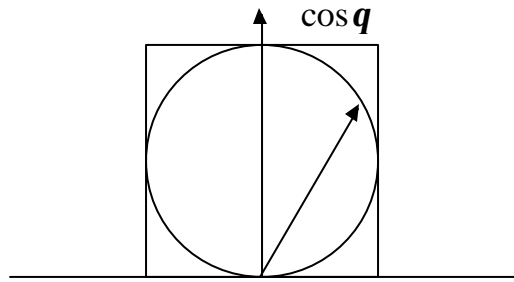
$$E = \int_{H^2} L_i(\mathbf{w}_i) \cos \mathbf{q}_i d\mathbf{w}_i$$

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Diffuse BRDF

Generate cosine weighted distribution

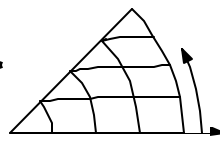
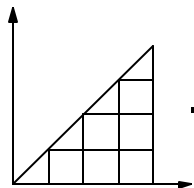
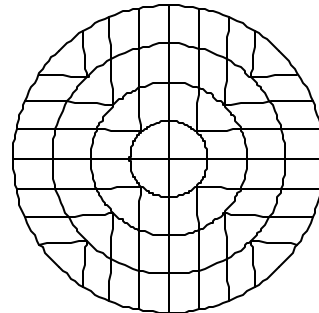
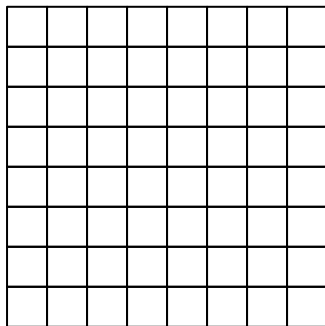


$$E = \int_{H^2} L_i(\mathbf{w}_i) \cos \mathbf{q}_i d\mathbf{w}_i$$

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Shirley's Mapping



$$r = U_1$$

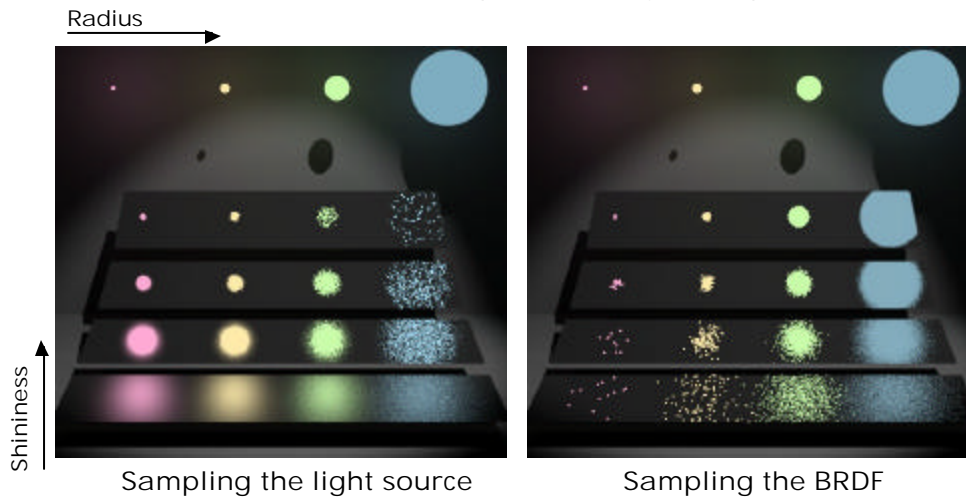
$$q = \frac{p U_2}{4 U_1}$$

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Multiple Importance Sampling

Reflection of a circular light source by a rough surface



Sampling the light source

Sampling the BRDF

$$\int f(x)g(x)dx$$

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Multiple Importance Sampling

Combine both sampling methods



From Veach and Guibas
Read Chapter 9, Veach

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Two sampling techniques

$$X_{1,i} \sim p_1(x) \quad X_{2,i} \sim p_2(x)$$

$$Y_{1,i} = \frac{f(X_{1,i})}{p_1(X_{1,i})} \quad Y_{2,i} = \frac{f(X_{2,i})}{p_2(X_{2,i})}$$

Weighted combination

$$Y_i = w_1 Y_{1,i} + w_2 Y_{2,i}$$

The balance heuristic

$$w_i(x) = \frac{p_i(x)}{p_1(x) + p_2(x)}$$

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Variance Reduction

Efficiency measure

$$\text{Efficiency} \propto \frac{1}{\text{Variance} \bullet \text{Cost}}$$

Some techniques

- Expected values vs. rejection sampling
- Importance sampling
- Estimators
- Sampling patterns: stratified, correlated, antithetic
- Russian roulette and splitting