Overview

Signal Processing

- Sampling and Reconstruction
- Aliasing and Antialiasing
- Nonuniform sampling: jittering and Poisson disk

Statistical Sampling

- Monte Carlo integration
- Sequential and adaptive sampling
- Discrepancy and Quasi-Monte Carlo
- Block design
- Distribution ray tracing

Probability Theory 1

Random variables

X is chosen by some random process

Probability distributions

$P(x) = \Pr(X_i < x)$ cumulative distribution function

$$p(x) = \frac{dP(x)}{dx}$$ probability density function

$$\Pr(a \le X_i \le b) = \int_a^b p(x)\,dx = P(b) - P(a)$$

Probability Theory 2

Expected values

Suppose $Y = f(X)$; then Y is also a random variable

$$E[Y] \equiv \int f(x)\, p(x)\, dx$$

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$
$$E[aY] = aE[Y]$$

Variance

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2 - 2YE[Y] + E[Y]^2]$$
$$= E[Y^2] - E[Y]^2$$

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$
$$V[aY] = a^2 V[Y]$$

Monte Carlo Integration

Integral

$$I(f) = \int_0^1 f(x)\, p(x)\, dx$$

Estimator

$$F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i = \frac{1}{N}\sum_{i=1}^{N} f(X_i)$$

$$E[F_N] = I(f)$$

Note: Use uniform pdf here, but may use any pdf.

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$
$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i]$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x)\, p(x)\, dx$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x)\, dx$$
$$= \int_0^1 f(x)\, dx$$

Convergence

Mean and standard deviation

$$m_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

$$E[m_N] = E[\frac{1}{N}\sum_i Y_i] = \frac{1}{N}\sum_i E[Y_i] = E[Y]$$

$$V[m_N] = V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N}V[Y]$$

$$s_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - m_N)^2$$

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Sequential Sampling

Central Limit Theorem

$$\lim_{N\to\infty} \Pr\left\{ m_N - E[Y] \le \frac{ts}{N^{1/2}} \right\} = \frac{1}{\sqrt{2p}} \int_{-\infty}^{t} e^{-x^2/2}\, dx$$

Sample until confidence in the estimate is high

- Student t-distribution

  Purgathofer

- Chi-squared distribution

  Lee, Redner, Uselton (1985)

Essentially the procedures used by pollsters

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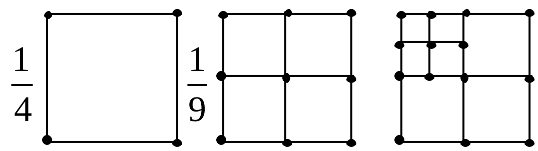
Adaptive Sampling

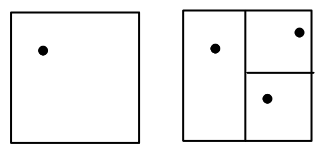
Whitted

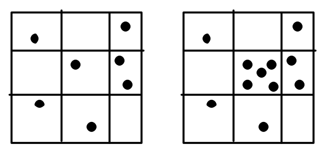
Recurse



Kajiya

Split node





Contrast

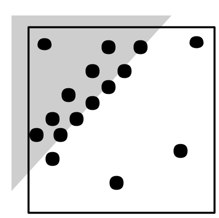
$$\frac{I_{Max} - I_{Min}}{I_{Max} + I_{Min}}$$

Mitchell

Shotgun blast

Other criteria ...

Filtering Non-uniform Samples



$$= \frac{1}{2}$$

1. Weight samples by "area"

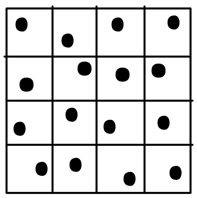
2. Multi-stage filter

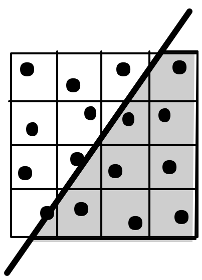
   Multiple box filters of ever increasing width

   $$k(x) = box(x) \otimes box(2x) \otimes box(2x) \otimes box(4x)$$

   See Figure 11-14 Mitchell (1991)

Stratified Sampling





Stratified sampling like jittered sampling

$$N = \frac{1}{N}\sum_{j=1}^{m} N_j \qquad F_N = \frac{1}{N}\sum_{j=1}^{m} N_j F_j$$

$$V[F_N] = \frac{1}{N}\sum_{j=1}^{m} N_j V[F_j]$$

Thus, if the variance in regions is less than the overall variance, there will be a reduction in resulting variance

For example: An edge through a pixel

$$V[F_N] = \frac{1}{N^2}\sum_{j=1}^{\sqrt{N}} V[F_j] = \frac{V[F_e]}{N^{1.5}}$$

High-dimensional Sampling

Stratified sampling (also numerical quadrature)

For a given error ...

$$E \sim \frac{1}{N^d}$$

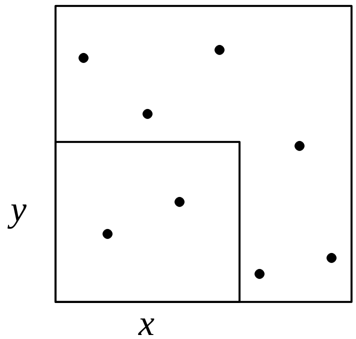
Random sampling

For a given variance ...

$$E \sim V^{1/2} \sim \frac{1}{N^{1/2}}$$

Monte Carlo much better for integration in high dimensional spaces

Discrepancy



$$\Delta(x,y) = \frac{n(x,y)}{N} - xy$$

$$A = xy$$

$n(x,y)$ number of samples in A

$$D_N = \max_{x,y} \left| \Delta(x,y) \right|$$

Theorem on Total Variation

Theorem: $$\left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \int f(x)\,dx \right| \le V(f)\, D_N$$

Proof: Integrate by parts

$$\frac{\partial \Delta(x)}{\partial x} = \frac{d(x - x_i)}{N} - 1$$

$$\int f(x)\left[\frac{d(x-x_i)}{N} - 1\right] dx$$

$$= \int f(x) \frac{\partial \Delta(x)}{\partial x} dx$$

$$= f\Delta\Big|_0^1 - \int \frac{\partial f(x)}{\partial x} \Delta(x)\, dx = -\int \frac{\partial f(x)}{\partial x} \Delta(x)\, dx$$

$$\le D_N \int \left| \frac{\partial f(x)}{\partial x} \right| dx = V(f) D_N$$

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer ***i*** in integer base ***b***

$$i = d_i \cdots d_2 d_1 d_0$$

$$f_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

$f_2(i)$

1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

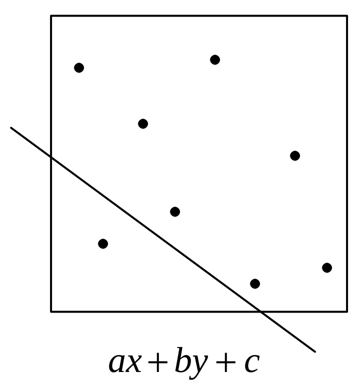
Hammersley points

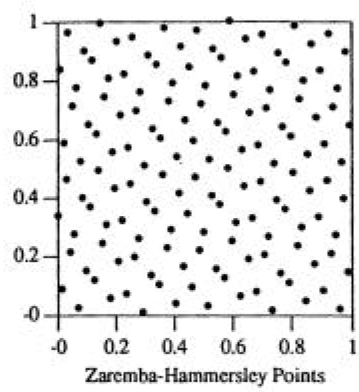
$$(i/N, f_2(i), f_3(i), f_5(i), \cdots) \qquad D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

Halton points (sequential)

$$(f_2(i), f_3(i), f_5(i), \cdots) \qquad D_N = O\left(\frac{\log^{d} N}{N}\right)$$

Edge Discrepancy





Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended
Story still unclear in higher dimensions?

Block Design

a	d	c	b
b	a	d	c
c	b	a	d
d	c	b	a

Latin Square

Alphabet of size n

Each symbol appears exactly once in each row and column

Improves discrepancy

a			
		a	
	a		
			a

N-Rook Pattern

$(p_x = \{1,2,3,4\}, p_y = \{4,2,3,1\})$

Incomplete block design

Replaced N^d samples with N samples

Permutations: $(p_1(i), p_2(i), \cdots p_d(i))$

Generalizations: N-queens, 2D projection

Distribution Ray Tracing

$$R = \int_t \iint_{\omega} \iint_A L(x,\omega,t)\, P(x)\, S(t) \cos\theta \; dA\, d\omega\, dt$$

Terms (5 dimensions)

- Pixel $\boldsymbol{x}$ – edge antialiasing
- Shutter $\boldsymbol{t}$ – motion blur
- Lens ω – depth of field

As we will see, even more integrals possible

- Reflections from surface
- Area light sources
- Number of bounces
- ...

Space-time Patterns

6	10	2	13
3	14	12	8
15	0	7	11
5	9	4	1

Cook Pattern

15	8	5	2
4	3	14	9
10	13	0	7
1	6	11	12

Pan-diagonal Magic Square

Fully populate ***(x,y)*** samples

- Recall blue noise good
  - Perceptually pleasing
  - Filtered during resampling
- Jitter to achieve blue noise

Distribute ***t*** samples

- Decorrelate space and time
- Nearby samples in space should differ greatly in time

Mitchell (1991) designs

Views of Sampling

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good
2. Statistical sampling
   - Monte Carlo: variance, central limit theorem
   - Adaptive sampling criteria, $N^{-1/2}$, high dimensional sampling
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Numerical
   - Quadrature/Integration rules
   - Smooth functions