Sampling and Reconstruction

The sampling and reconstruction process

- Real world: continuous
- Digital world: discrete

Basic signal processing

- Fourier transforms
- The convolution theorem
- The sampling theorem

Aliasing and antialiasing

- Uniform supersampling
- Nonuniform supersampling

Imagers = Signal Sampling

All imagers convert a continuous image to a discrete sampled image by integrating over the active “area” of a sensor.

$$R = \int_t \iint_w \iint_A L(x,w,t)\, P(x)\, S(t) \cos q \, dA\, dw\, dt$$

Examples:

- Retina: photoreceptors
- CCD array
- Vidicon: phosphors

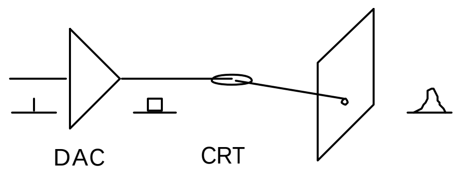
Virtual “computer graphics” cameras do not integrate, instead they simply sample ...

Displays = Signal Reconstruction

All physical displays recreate a continuous image from a discrete sampled image by using a finite sized source of light for each pixel.

Examples:

- DAC's: sample and hold
- Cathode ray tube: phosphor spot and grid



Sampling in Computer Graphics

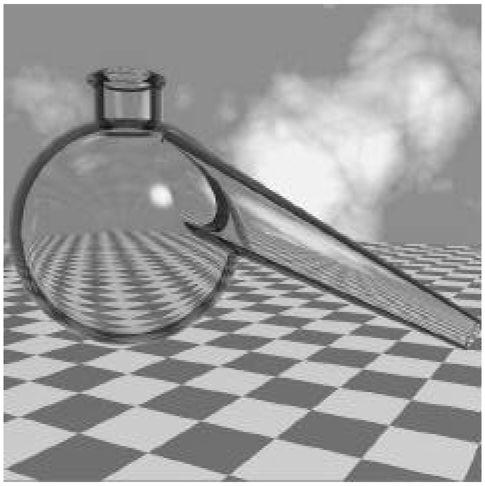
Artifacts due to sampling - Aliasing

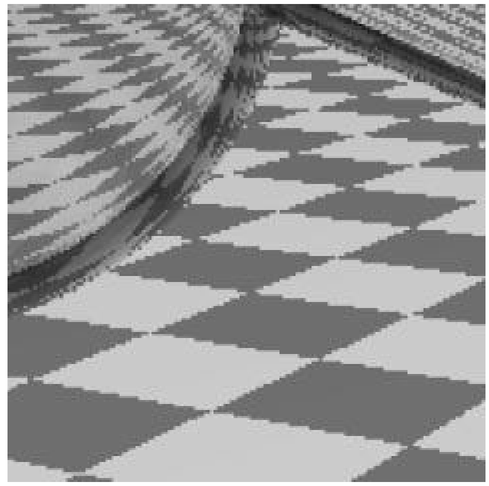
- Jaggies
- Moire
- Flickering small objects
- Sparkling highlights
- Temporal strobing

Preventing these artifacts - Antialiasing

Jaggies

Retort sequence by Don Mitchell





Staircase pattern or jaggies

Spectral Analysis / Fourier Transforms

Spectral representation treats the function as a weighted sum of sines and cosines

Each function has two representations

- Spatial (time) domain - normal representation
- Frequency domain - spectral representation

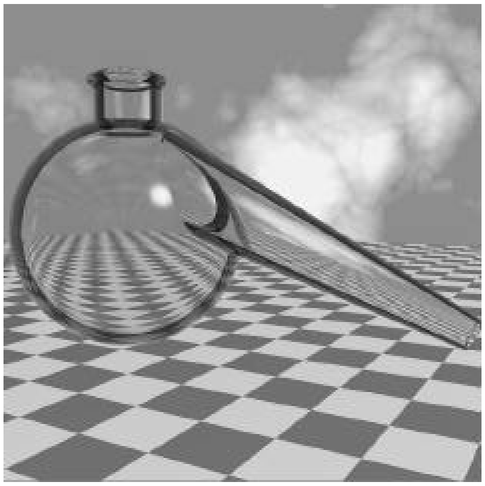
The *Fourier transform* converts between the spatial and frequency domain

Spatial Domain ⇒ $F(w) = \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$ ⇒ Frequency Domain

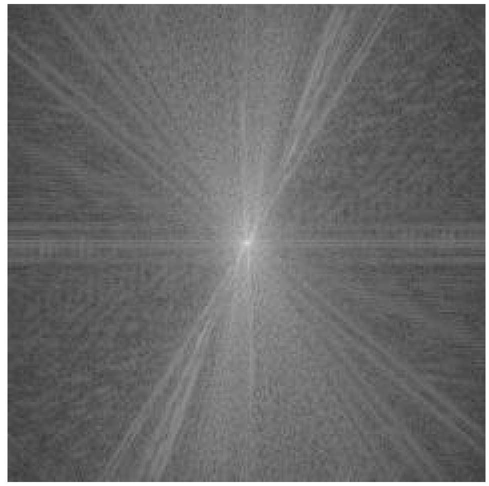
Spatial Domain ⇐ $f(x) = \frac{1}{2p} \int_{-\infty}^{\infty} F(w) e^{iwx}\, dw$ ⇐ Frequency Domain

Spatial and Frequency Domain

Spatial Domain



Frequency Domain



Convolution

Definition

$$h(x) = f \otimes g = \int f(x')g(x-x')dx'$$

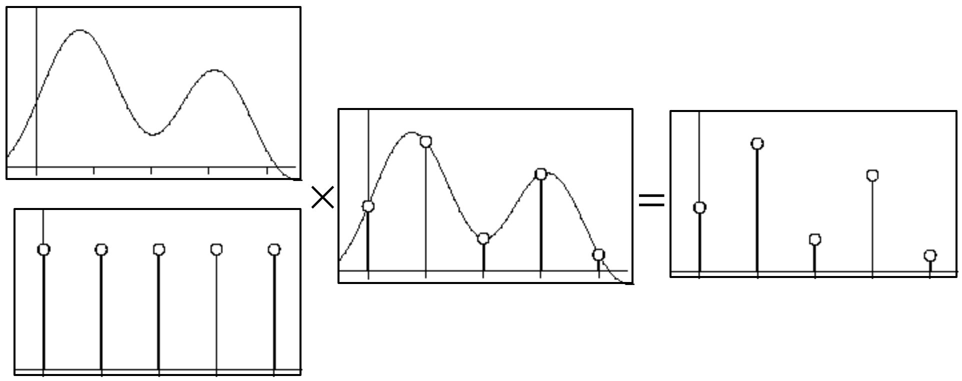
Convolution Theorem: Multiplication in the frequency domain is equivalent to convolution in the space domain.

$$f \otimes g \leftrightarrow F \times G$$

Symmetric Theorem: Multiplication in the space domain is equivalent to convolution in the frequency domain.

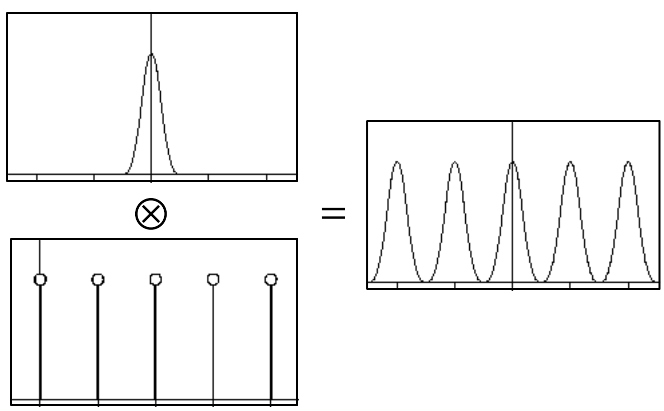
$$f \times g \leftrightarrow F \otimes G$$

Sampling: Spatial Domain



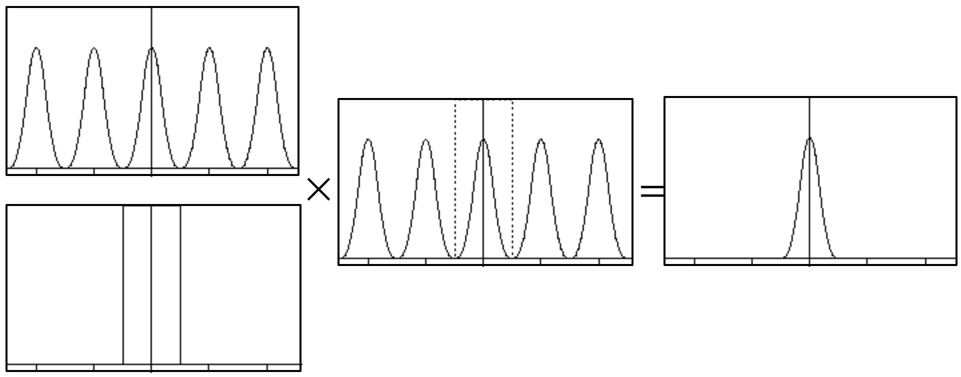
$$\mathrm{III}(x) = \sum_{n=-\infty}^{n=\infty} d\,(x - nT) \qquad f_s = \frac{1}{T} \quad w_s = 2pf_s = \frac{2p}{T}$$

Sampling: Frequency Domain



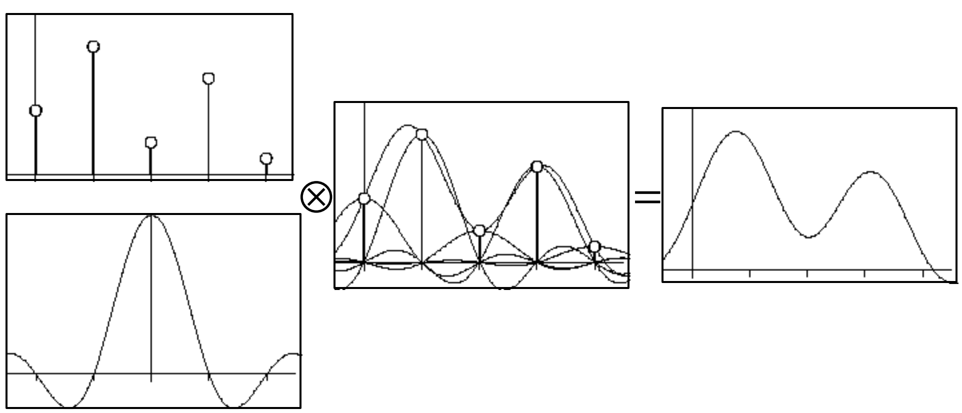
$$\mathrm{III}(w) = \sum_{n=-\infty}^{n=\infty} d\,(w - nw_s)$$

Reconstruction: Frequency Domain



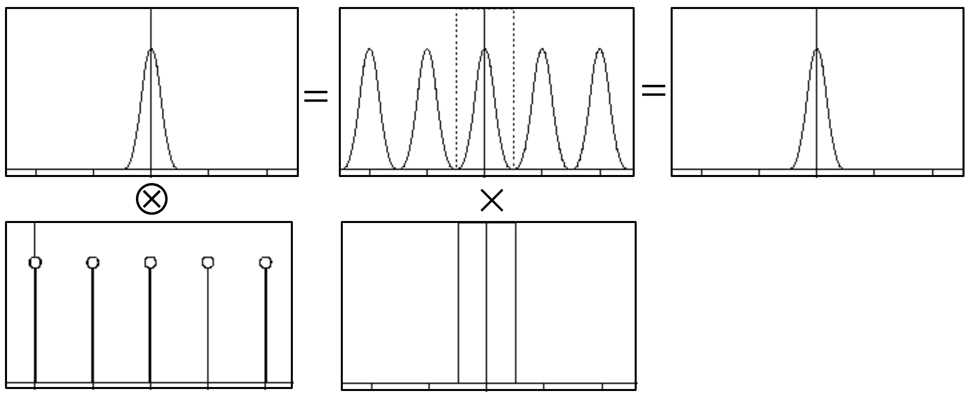
$$\mathrm{II}(x) = \begin{cases} 1 & |x| \le \tfrac{1}{2} \\ 0 & |x| > \tfrac{1}{2} \end{cases}$$

Reconstruction: Spatial Domain



$$\operatorname{sinc} x = \frac{\sin px}{px}$$

Sampling and Reconstruction



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Sampling Theorem

This result if known as the Sampling Theorem and is due to Claude Shannon who first discovered it in 1949

> A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the sampling frequency

For a given bandlimited function, the rate at which it must be sampled is called the Nyquist Frequency

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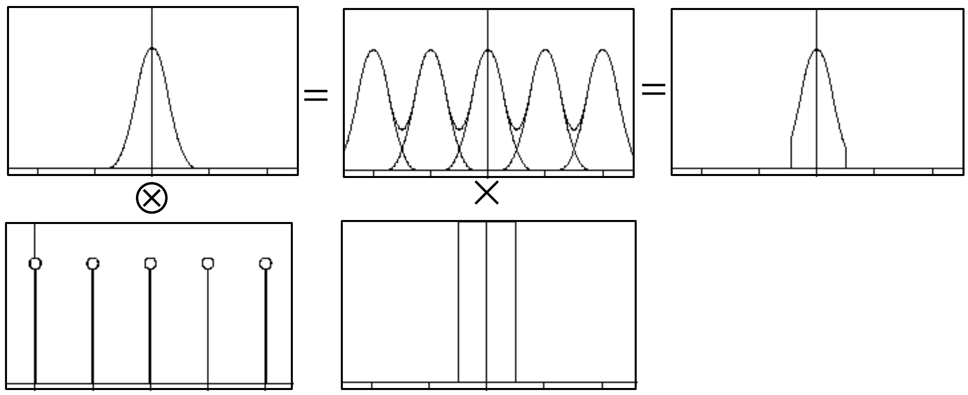
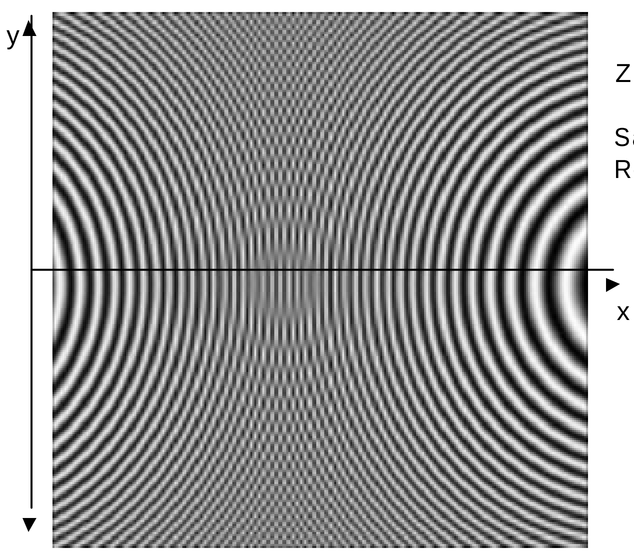
Undersampling: Aliasing



Sampling a "Zone Plate"



Zone plate: $\sin x^2 + y^2$

Sampled at 128x128
Reconstructed to 512x512

Left rings: part of signal
Right rings: prealiasing
Middle rings: postaliasing

Ideal Reconstruction

Ideally, use a perfect low-pass filter - the sinc function - to bandlimit the sampled signal and thus remove all copies of the spectra introduced by sampling

Unfortunately,

- The sinc has infinite extent and we must use simpler filters with finite extents. Physical processes in particular do not reconstruct with sincs
- The sinc may introduce ringing which are perceptually objectionable

Mitchell Cubic Filter

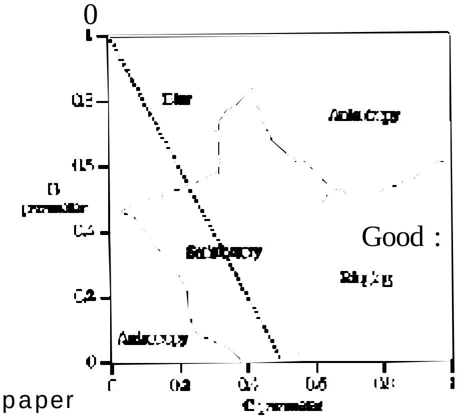
$$h(x) = \frac{1}{6}\begin{cases} (12-9B-6C)x^3 + (-18+12B+6C)x^2 + (6-2B) & |x|<1 \\ (-B-6C)x^3 + (6B+30C)x^2 + (-12B-48C)x + (8B+24C) & 1<|x|<2 \\ 0 & otherwise \end{cases}$$

Properties:

$$\sum_{n=-\infty}^{n=\infty} h(x) = 1$$

B - spline : (1,0)

Catmull - Rom : (0,1/2)



Good : (1/3,1/3)

From Mitchell and Netravali
Look at other figures in that paper

Aliasing and Antialiasing

If the scene contains frequencies greater than the Nyquist Frequency, then we have a aliasing problem

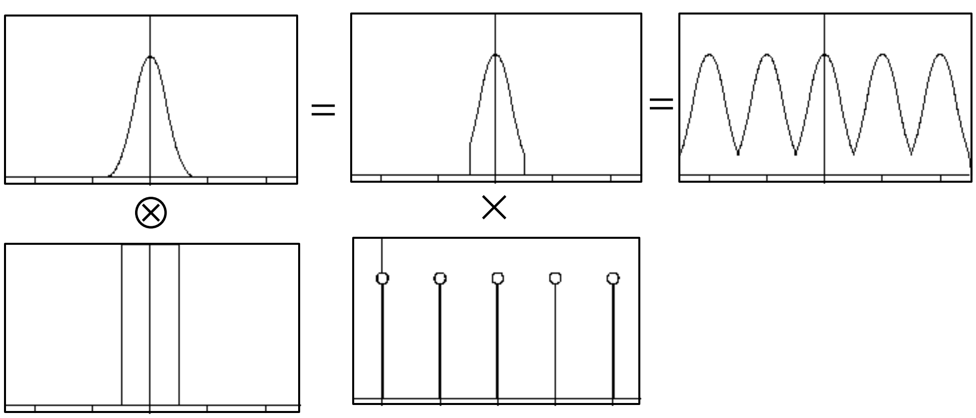
Results of aliasing:

- Jaggies
- Moire
- Flickering small objects
- Sparkling highlights
- Temporal strobing

Preventing aliasing or antialiasing:

1. Analytically prefilter the signal
2. Uniform supersampling and resample
3. Nonuniform or stochastic sampling

Antialiasing by Prefiltering



Antialiasing by Prefiltering

Ideally, low-pass with a perfect filter (a sinc function) to bandlimit the function to the Nyquist sampling rate.

Unfortunately, the sinc has infinite extent and we must use simpler filters (like a box filter, or area average).

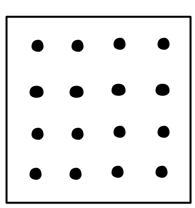
Practically:

- Constant colored polygonal fragments doable
- Complex environments not doable

Uniform Supersampling

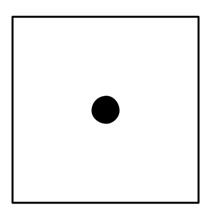
Increasing the sampling rate moves each copy of the spectra further apart, potentially reducing the overlap and thus aliasing

Resulting samples must be resampled (filtered) to image sampling rate



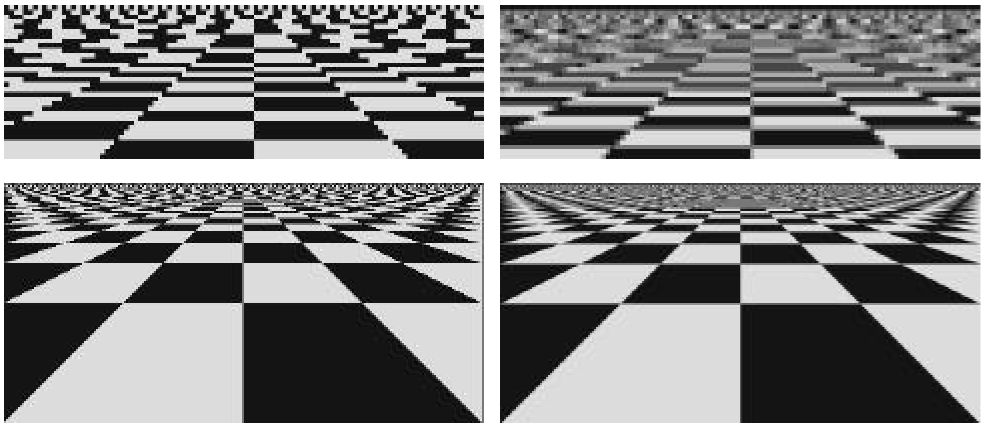
Samples

$$Pixel = \sum_s w_s \cdot Sample_s$$



Pixel

Point vs. Supersampled



Point | 4x4 Supersampled

Checkerboard sequence by Tom Duff

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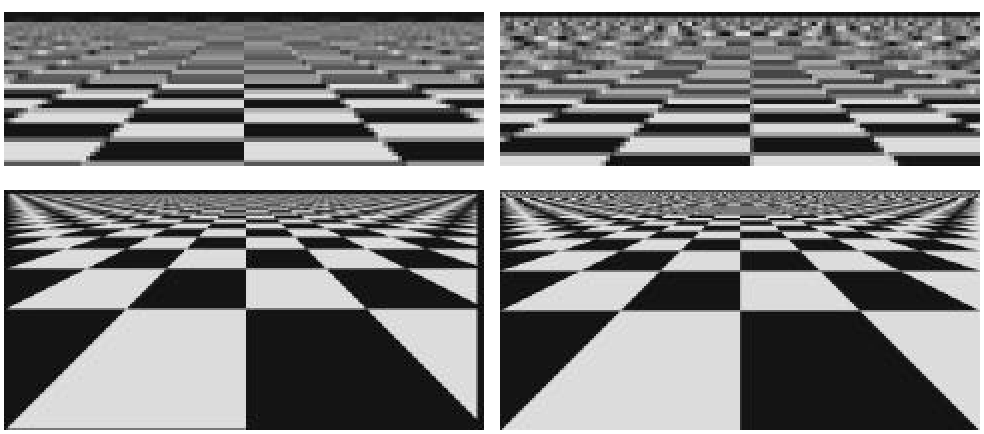
Analytic vs. Supersampled



Exact Area | 4x4 Supersampled

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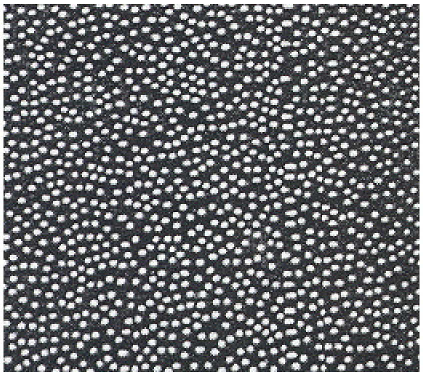
Distribution of Extrafoveal Cones



Monkey eye cone distribution

Fourier transform

Yellot theory

- Aliases replaced by noise
- Visual system less sensitive to high frequency noise

Non-uniform Sampling

Intuition

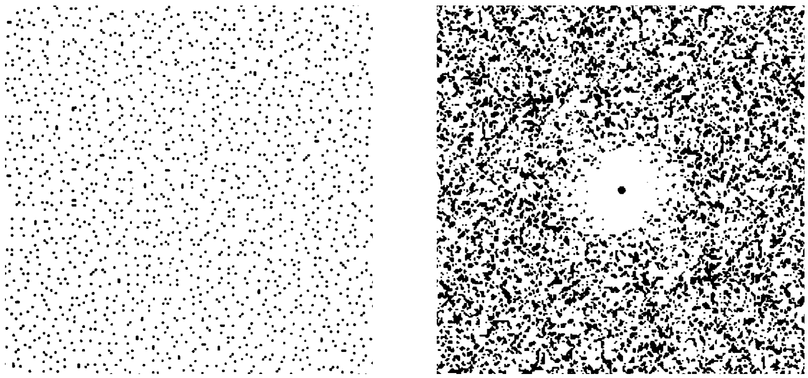
Uniform sampling

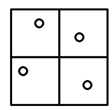
- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
- Aliases are coherent, and very noticable

Non-uniform sampling

- Samples at non-uniform locations have a different spectrum; a single spike plus noise
- Sampling a signal in this way converts aliases into broadband noise
- Noise is incoherent, and much less objectionable

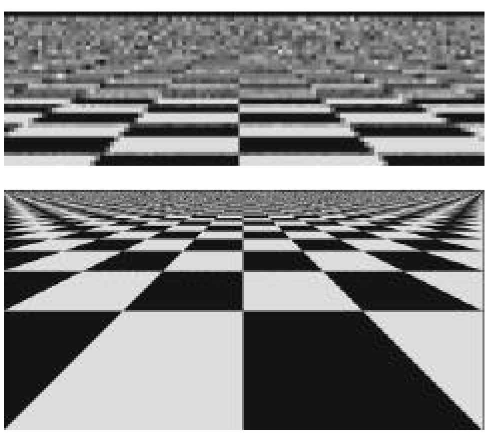
Jittered Sampling



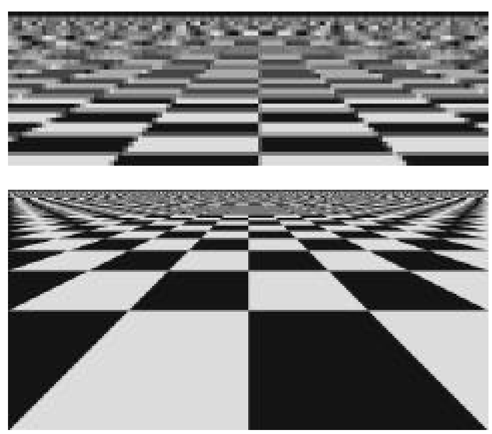


Add uniform random jitter to each sample

Jittered vs. Uniform Supersampling



Jittered Sampling



4x4 Uniform

Analysis of Jitter

Non-uniform sampling

$$s(x) = \sum_{n=-\infty}^{n=\infty} d(x - x_n)$$

$$x_n = nT + j_n$$

Jittered sampling

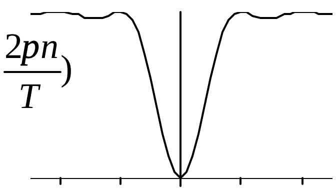
$$j_n \sim j(x)$$

$$j(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$J(w) = \operatorname{sinc} w$$

$$S(w) = \frac{1}{T}\left[1 - |J(w)|^2\right] + \frac{2p}{T^2}|J(w)|^2 \sum_{n=-\infty}^{n=-\infty} d(w - \frac{2pn}{T})$$

$$= \frac{1}{T}\left[1 - \operatorname{sinc}^2 w\right] + d(w)$$



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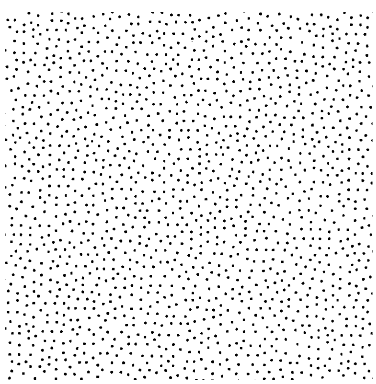
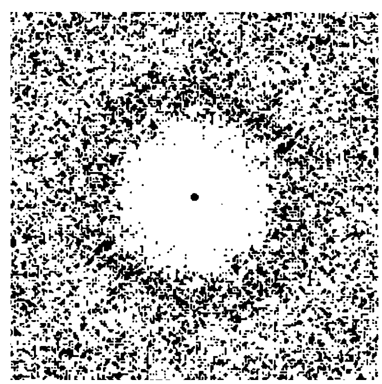
Poisson Disk Sampling





Dart throwing algorithm

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Filtering Nonuniform Samples

1. Reconstruction

$$P = \frac{\sum_s w(x_s, y_s) \cdot S(x_s, y_s)}{\sum_s w(x_s, y_s)}$$

Note: For jittered samples may approximate with filter coefficients at centers of each cell

2. Filtering further attenuates high frequency noise
3. Problematical if samples are clumpy

Tiling